A. H. Clifford and S. MacLane [2] considered in 1941 the group of factor-sets $H^2(\Gamma, U)$ of a finite group $\Gamma$ over its abstract unit group $U$. They proved the main theorem to the effect that $H^2(\Gamma, U)$ is isomorphic to the multiplicator $M$ of $\Gamma$ defined by I. Schur and also several other theorems under the assumption that $\Gamma$ is a solvable group. They conjectured that these should hold for general finite groups $\Gamma$. In 1942 A. Weil proved the main theorem for general finite groups $\Gamma$, but this result was not published. In this short note we shall prove that all the theorems in [2] are valid for general finite groups $\Gamma$, and also we shall extend their results for all (positive, zero, and negative) dimensional cohomology groups.

1. We shall first prove a general lemma on cohomology groups. Let $\Delta$ be a finite group, and let $E$ be a $\Delta$-module. Suppose that $A_1, A_2$ are two $\Delta$-submodules which are disjoint: $A_1 \cap A_2 = 0$. Then we have the following commutative diagram such that each row and each column are exact:

$$
\begin{array}{ccc}
0 & \rightarrow & A_1 \\
\downarrow & & \downarrow \jmath_{21} \\
0 & \rightarrow & (A_1 + A_2)/A_2 \rightarrow 0 \\
\downarrow & & \downarrow \iota_{12} \\
0 & \rightarrow & A_2 \\
\downarrow j_{11} & \rightarrow & E \\
\downarrow \jmath_{12} & \rightarrow & E/A_2 \rightarrow 0 \\
\downarrow \iota_{13} & & \\
0 & \rightarrow & (A_1 + A_2)/A_1 \\
\downarrow j_{12} & \rightarrow & E/A_1 \\
\downarrow \jmath_{23} & \rightarrow & E/(A_1 + A_2) \rightarrow 0 \\
\downarrow & & \\
0 & \rightarrow & 0 \\
\end{array}
$$

Let us denote, in general, by $H^r(\Delta, A)$ the $r$-cohomology group of a group $\Delta$ over a $\Delta$-module $A$.

**Lemma.** Assume that $H^r(\Delta, E) = 0$ for $r = 0, \pm 1, \pm 2, \ldots$. Then we have for all $r = 0, \pm 1, \pm 2, \ldots$,

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1 The author wishes to express his thanks to Professor Saunders MacLane who kindly told him this fact and allowed him to read an unpublished manuscript about it.

2 For the definition of negative dimensional cohomology groups of a finite group and for the properties of cohomology groups see, for example, Artin-Tate [1].
\[(I)\] \ \ H^r(\Delta, E/A_1) \cong H^{r+1}(\Delta, A_1) \cong H^{r+1}(\Delta, (A_1 + A_2)/A_2),
\]

\[(II)\] \ \ 0 \to H^r(\Delta, E/A_2) \xrightarrow{j^*} H^r(\Delta, E/(A_1 + A_2)) \xrightarrow{\delta} H^{r+1}(\Delta, (A_1 + A_2)/A_2) \xrightarrow{i^*} 0 \text{ (exact)}

and similar formulas hold by interchanging the subscripts 1 and 2.

\[(III)\] \ \ H^r(\Delta, E/(A_1 + A_2)) = j_{23}^*(H^r(\Delta, E/A_1)) + j_{13}^*(H^r(\Delta, E/A_2)).

**Proof.** (i) \((I)\) is evident by our assumption \(H^r(\Delta, E)=0\). (ii) Since \((i_{13})^* = (j_{23})^* \circ (i_{12})^* \circ (j_{21})^{-1}\) and \((i_{12})^* = 0\) by our assumption, we have \((i_{13})^* = 0\). Hence we get \((II)\) by the exact sequence of cohomology groups derived from the 3rd column of the diagram (1). (iii) From (1) follows

\[(2)\] \ \ 0 \to H^r(\Delta, E/A_1) \xrightarrow{j_{23}} H^r(\Delta, E/(A_1 + A_2)) \xrightarrow{\delta_2} H^{r+1}(\Delta, (A_1 + A_2)/A_1) \xrightarrow{i_{23}^*} 0.

Here \(j_{23}\) is an into-isomorphism and \(\delta_2, j_{11}^*\) are onto-isomorphisms. Hence \(j_{13}^*(H^r(\Delta, E/A_2))\) is a splitting system of representatives of \(H^r(\Delta, E/(A_1 + A_2)) \mod j_{23}^*(H^r(\Delta, E/A_1))\). This proves \((III)\), q.e.d.

2. Let \(\Gamma\) be a finite group of order \(n\), and \(\Gamma(Z)\) be its group ring over the integers \(Z\). Put \(u = \sum_{\sigma \in \Gamma} \sigma \in \Gamma(Z)\). Then by definition the factor group \(U = \Gamma(Z)/zu\) is the abstract unit group of \(\Gamma\). Now let \(\Delta\) be an arbitrary subgroup of \(\Gamma\). Let us take \(E = \Gamma(Z), A_1 = \sum_{\sigma \neq 1} Z(1 - \sigma), \text{and} A_2 = Zu\). Clearly \(A_1 \cap A_2 = 0\). Since \(E = \Gamma(Z)\) is \(\Delta\)-free, the assumption in the lemma is satisfied. Hence we can apply the lemma. Here we may identify \(E/A_1 = Z\) and \(j_{12} = \text{tr}\), where \(\text{tr} (\sum_{\sigma} a_{\sigma} \cdot \sigma) = \sum_{\sigma} a_{\sigma} \in Z (a_{\sigma} \in Z)\). Then the 3rd row of the diagram (1) may be replaced by

\[(3)\] \ \ 0 \to Zu \xrightarrow{i_{23}} Z \xrightarrow{j_{23}} Z/nZ \to 0

where \(\Delta\) operates on these modules trivially. Also \(j_{18}\) and \(j_{11}\) become the homomorphism \(\text{tr}\) induced in \(U \to Z/nZ\) and \(Zu \to Z/nZ\) respectively. Finally put \(U_0 = (A_1 + A_2)/A_2\), which is the kernel of the mapping \(\text{tr} U \to Z/nZ\). By these substitutions we have the following formulas from our lemma:

For all \(r = 0, \pm, \pm 2, \cdots\)
(I) \[ H^r(\Delta, U) \cong H^{r+1}(\Delta, Z), \]

(II) \[ H^{r-1}(\Delta, Z)^{j_2^* \cdot \delta} \cong H^r(\Delta, U_0), \]

(II) \[ 0 \to H^r(\Delta, U) \to H^r(\Delta, Z/nZ) \to H^{r+1}(\Delta, U_0) \to 0 \text{ (exact)}, \]

(II) \[ 0 \to H^r(\Delta, Z) \to H^r(\Delta, Z/nZ) \to H^{r+1}(\Delta, nZ) \to 0 \text{ (exact)}, \]

(III) \[ H^*(\Delta, Z/nZ) = j_2^*(H^*(\Delta, Z)) + tr^*(H^*(\Delta, U)), \]

where \( \Delta \) operates trivially on \( Z, nZ \) and \( Z/nZ \).

Now we get several theorems in [2] as corollaries of these formulas. Namely, from (I) follows

(i) \( H^0(\Delta, U_0) = H^1(\Delta, Z) = 0; \) \( H^1(\Delta, U_0) = H^0(\Delta, Z) \cong Z/mZ \) where \( m \) is the order of \( \Delta \); \( H^2(\Delta, U_0) = H^1(\Delta, Z) = 0 \) (formulas (1), (2) in §6 and corollary in §1 of [2]). From (II) follows

(ii) \( tr^*: H^2(\Delta, U) \to H^2(\Delta, Z/nZ) \) is an into-isomorphism (Theorem 1.A of [2]),

(iii) \( i^*(H^1(\Delta, U_0)) = 0 \) in \( H^1(\Delta, U) \) (Theorem 1.B of [2]). From (II) and (II) follows

(iv) \( tr^*: H^1(\Delta, U) \to H^1(\Delta, Z/nZ) \) is an onto-isomorphism (Theorem 2.B of [2]).

3. Let \( \Omega \) be an algebraically closed field of characteristic not dividing the order \( n \) of \( \Gamma \). Then the multiplicator \( M \) of \( \Gamma \) is defined by I. Schur as \( M = H^2(\Gamma, \Omega^*) \), where \( \Gamma \) acts trivially on the multiplicative group \( \Omega^* \). Let \( W \) be the group of all the roots of unity in \( \Omega \). Consider the exact sequence \( 1 \to W \to \Omega^* \to \Omega^*/W \to 1 \). Since the group \( \Omega^*/W \) is uniquely divisible, \( H^r(\Delta, \Omega^*/W) = 0 \) for all \( r \). Hence we have \( M = H^2(\Gamma, \Omega^*) \cong H^2(\Gamma, W) \cong H^2(\Delta, Q/Z) \), where \( Q \) is the additive group of rationals. Let the homomorphism \( \text{aver.} \) be defined on \( \Gamma(Z) \) by

\[ \text{aver.} \left( \sum_\sigma a_\sigma \cdot \sigma \right) = \frac{1}{n} \sum_\sigma a_\sigma = \frac{1}{n} \text{tr} \left( \sum_\sigma a_\sigma \cdot \sigma \right) \in Q. \]

This homomorphism \( \text{aver.} \) induces also the homomorphism \( \text{aver.}*: H^2(\Gamma, U) \to H^2(\Gamma, Q/Z) \) is an onto-isomorphism. For the sake of completeness we shall give here a proof which is essentially the same as that of A. Weil. Let us consider the commutative diagram:
0 → \mathbb{Z}u → \Gamma(Z) → U → 0

(4) \downarrow \phi_1 \downarrow \phi_2 \downarrow \phi_3 \\
0 → Z → Q → Q/\mathbb{Z} → 0

where \phi = \text{aver}. Since \text{H}^r(\Delta, Q) = \text{H}^r(\Delta, \Gamma(Z)) = 0$ for all $r$, this diagram induces the commutative diagram:

0 → \text{H}^r(\Delta, U) \rightarrow \text{H}^{r+1}(\Delta, \mathbb{Z}u) → 0

(5) \downarrow \phi_1^* \downarrow \phi_2^* \\
0 → \text{H}^r(\Delta, Q/\mathbb{Z}) \rightarrow \text{H}^{r+1}(\Delta, \mathbb{Z}) → 0.

Here $\phi_1^*$ is an onto-isomorphism, so is $\phi_2^* = \delta_2^{-1} \circ \phi_1^* \circ \delta_1$. Hence we get

(IV) aver. * : $\text{H}^r(\Delta, U) → \text{H}^r(\Delta, Q/\mathbb{Z})$ is an onto-isomorphism for all $r$.

The relation between $\text{tr}^*$ and $\text{aver.}^*$ on $\text{H}^r(\Delta, U)$ is given as follows. Let $\psi$ be the homomorphism defined by $\psi(\alpha) = \alpha \times (1/n)$ on $Z(\rightarrow Q)$, $nZ(\rightarrow Z)$ and $\mathbb{Z}/n\mathbb{Z}(\rightarrow Q/\mathbb{Z})$ respectively. Then we have $\text{aver.} = \psi \circ \text{tr}$, and $\psi$ induces the homomorphism: $\text{H}^r(\Delta, \mathbb{Z}/n\mathbb{Z})^* \rightarrow \text{H}^r(\Delta, Q/\mathbb{Z})$. Then we have

(V) $\psi^* \circ j_{12}^*(\text{H}^r(\Delta, Z)) = 0$ and $\psi^*$ is an isomorphism of $\text{tr}^*(\text{H}^r(\Delta, U))$ onto $\text{H}^r(\Delta, Q/\mathbb{Z})$.

**Proof.** Let us consider the commutative diagram:

0 → $n\mathbb{Z} \rightarrow Z \rightarrow Z/n\mathbb{Z} → 0$

\[ \downarrow \psi_1 \downarrow \psi_2 \downarrow \psi_3 \]

0 → $Z \rightarrow Q \rightarrow Q/\mathbb{Z} → 0$.

Since $(\psi_2)^* = 0$ by $\text{H}^r(\Delta, Q) = 0$, we have $\psi_3^* \circ j_{12}^* \circ \psi_3^* = 0$. Since $\phi_3^* = \psi_3^* \circ j_{12}^*$ and $\phi_3^* (j_{12}^*)$ is an isomorphism-onto (-into), so $\psi_3^*$ is an onto-isomorphism, q.e.d. These considerations actually cover Theorem 2.A of [2].

**References**


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