ON THE RADIAL LIMITS OF ANALYTIC FUNCTIONS

A. J. LOHWATER

1. Examples have been given [5, p. 185] of functions \(f(z)\), analytic in the unit-circle \(K: |z| < 1\), and not identically constant, for which the radial limit \(f(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})\) is zero for all \(e^{i\theta}\) on \(|z| = 1\) except for a set of linear measure zero. In view of the Riesz-Nevanlinna theorem [6, p. 197], such functions cannot be bounded, or even of bounded characteristic, in \(|z| < 1\). Functions of this sort appear again whenever we have an analytic function \(f(z)\) whose radial limits coincide almost everywhere with the radial limits of a bounded analytic function \(g(z)\), for the difference \(F(z) = f(z) - g(z)\) has a radial limit zero almost everywhere on \(|z| = 1\). The Riesz-Nevanlinna theorem shows that, if \(f(z)\) is bounded, or of bounded characteristic, and if the radial limit values of \(f(z)\) coincide almost everywhere on an arc of \(|z| = 1\) with the radial limit values of \(g(z)\), then \(F(z)\) must be identically zero in \(|z| < 1\). The object of this note is to discuss certain aspects of the behavior of nonconstant analytic functions whose radial limits vanish almost everywhere on an arc \(A(\theta_1 < \theta < \theta_2)\) of \(|z| = 1\). One result of such a study (which the author plans as a sequel to this note) will be to give some idea of the way in which a function \(f(z)\), whose radial limits coincide almost everywhere with the radial limits of a function \(g(z)\) of bounded characteristic, can differ from \(g(z)\).

We shall say that a nonconstant function \(f(z)\), analytic in \(|z| < 1\), is of class \((LP)\) on an arc \(A\) of \(|z| = 1\), if \(\lim_{r \to 1^-} f(re^{i\theta}) = f(e^{i\theta}) = 0\) for almost all \(e^{i\theta}\) belonging to the arc \(A\). If the arc \(A\) is the whole circumference \(|z| = 1\), we shall say simply that the function \(f(z)\) is of class \((LP)\).

One property of functions which are of class \((LP)\) on an arc \(A\) is immediate: the cluster set of \(f(z)\) at each point \(e^{i\theta_0}\) of \(A\) (i.e., the set of all values \(\alpha\) with the property that there exists a sequence \(\{z_n\}, |z_n| < 1, \lim_{n \to \infty} z_n = e^{i\theta_0}\), such that \(\lim_{n \to \infty} f(z_n) = \alpha\) is the whole plane. For, if there is a point \(e^{i\theta_0}\) on \(A\) and a complex number \(\alpha\) which does not belong to the cluster set of \(f(z)\) at \(e^{i\theta_0}\), then there is a circular neighborhood \(V(e^{i\theta_0})\) of \(e^{i\theta_0}\) such that, in \(V(e^{i\theta_0}) \cap K\), the function \(g(z) = [f(z) - \alpha]^{-1}\) is analytic and bounded. Since the function \(g(z)\) has the constant limit \(-1/\alpha\) along almost all normal segments drawn to that arc of \(|z| = 1\) which bounds \(V \cap K\), it follows from a simple corollary of the Riesz-Nevanlinna theorem that \(g(z)\), and hence \(f(z)\),

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must be identically constant in $V \cap K$ and, a fortiori, in $|z| < 1$. This property, i.e., that the cluster set is the whole plane, sometimes called the Weierstrass property, suggests that we investigate the values $\xi$ which $f(z)$ admits as asymptotic values, i.e., the values to which $f(z)$ tends as $z$ approaches a point $P$ of $|z| = 1$ along a curve terminating at $P$. We shall show (Theorem 1) that every complex value $\xi$ (including $\infty$) is an asymptotic value of a function $f(z)$ of class (LP) provided that the $\xi$-points, i.e., the points $z_k$ for which $f(z_k) = \xi$, satisfy the condition

$$\sum_{k=1}^{\infty} (1 - |z_k|) < \infty.$$  

In Theorem 2 we show that a function of class (LP) on an arc $A$ admits every complex number $\xi$ as an asymptotic value in every neighborhood of every point $e^{i\theta}$ of $A$ if the $\xi$-points in some neighborhood $V(e^{i\theta}) \cap K$ of $e^{i\theta}$ satisfy (1). Theorem 2 then contains Theorem 1, but the proof of Theorem 1 is considerably simpler, and we give a separate proof.

**Lemma 1.** Let $f(z)$ be analytic and different from 0 in $|z| < 1$, and let the modulus $|f(re^{i\theta})|$ have radial limit 1 for almost all $e^{i\theta}$ on $|z| = 1$. Then unless $f(z)$ is identically constant in $|z| < 1$, there exists a Jordan arc $\mathcal{L}$, lying in $|z| < 1$ and terminating at a point $e^{i\theta_0}$ of $|z| = 1$, such that, as $z \to e^{i\theta_0}$ along $\mathcal{L}$, either $f(z) \to 0$ or $f(z) \to \infty$. If there exists no path along which $f(z) \to 0$, then $|f(z)| > 1$ in $|z| < 1$.

Lemma 1 is equivalent to Theorems 5 and 6 of [3], and its proof is omitted here. For brevity, we shall say that a function which is analytic in $|z| < 1$ and whose modulus $|f(re^{i\theta})|$ has radial limit 1 for almost all $e^{i\theta}$ on $|z| = 1$ will be called of class (U) in $|z| < 1$.

**Theorem 1.** If $f(z)$ is analytic in $|z| < 1$ and of class (LP), then every complex number (including $\infty$) which satisfies (1) is an asymptotic value of $f(z)$.

Assume that a finite $\xi$ satisfying (1) is not an asymptotic value of $f(z)$; clearly, we need not consider the case that $\xi = 0$. Since $f(z)$ is of class (LP), the function $\phi(z) = \xi^{-1}[\xi - f(z)]$ has radial limit 1 almost everywhere and is then of class (U) in $|z| < 1$. Because the $\xi$-points of $f(z)$ satisfy (1), we may write $\phi(z) = B_1(z)F(z)$, where $B_1(z)$ is a Blaschke product extended over the zeros of $\phi(z)$. It is well known [7, p. 94] that the radial limits of a Blaschke product exist and have modulus 1 almost everywhere on $|z| = 1$. From this it follows that $F(z)$ is of class (U) without zeros in $|z| < 1$. It is then a conse-
quence of Lemma 1 that, unless \( F(z) \) is identically constant, \( F(z) \) must admit either 0 or \( \infty \) as an asymptotic value. We remark first that \( F(z) \) cannot be constant; for if \( \phi(z) \) reduces to a Blaschke product whose radial limit is 1 almost everywhere, the Riesz-Nevanlinna theorem shows that \( \phi(z) \), and consequently \( f(z) \), is constant. We assert next that 0 must be an asymptotic value of \( F(z) \); otherwise \( |F(z)| > 1 \) in \( |z| < 1 \), so that \( \phi(z) \) could be expressed as the quotient of two bounded functions in \( |z| < 1 \), i.e., \( \phi(z) \) would be of bounded characteristic in \( |z| < 1 \). Again, by the Riesz-Nevanlinna theorem, \( \phi(z) \) would be constant in \( |z| < 1 \). Since zero must now be an asymptotic value of \( F(z) \), and since \( B_1(z) \) is bounded, \( \phi(z) \) must admit zero as an asymptotic value, so that \( 1 \) is an asymptotic value of \( f(z) \).

To show that \( f(z) \) admits \( \infty \) as an asymptotic value,\(^1\) we remark that the function \( g(z) = e^{\phi(z)} \) is of class (U) without zeros in \( |z| < 1 \). Applying Lemma 1 to \( g(z) \), we see that, since \( g(z) \) is not constant, \( g(z) \) admits either 0 or \( \infty \) as an asymptotic value, so that there exists at least one path \( \mathcal{L} \) terminating at some point \( e^{i\theta_0} \) of \( |z| = 1 \) along which \( f(z) \to \infty \). Thus Theorem 1 is proved.

We remark that Theorem 1 is related to a recent result of Cartwright and Collingwood [2, p. 112], the added hypothesis that \( f(z) \) be of class (LP) in \( |z| < 1 \) allowing us to obtain a stronger conclusion to part of Theorem 9 of their paper.

2. In order to determine how frequently a function of class (LP) admits as an asymptotic value a complex number \( \xi \) satisfying (1), it will be necessary to use a form of the Schwarz reflection principle developed recently in [4]. We summarize this principle as a lemma.

**Lemma 2.** Let \( f(z) \) be meromorphic in \( |z| < 1 \) and let \( A \) be the arc \( 0 \leq \theta_1 < \theta < \theta_2 < 2\pi \). Let there exist an \( \varepsilon > 0 \) such that \( f(z) \) has no zeros or poles in the region \( 0 < 1 - |z| < \varepsilon \), \( \theta_1 < \arg z < \theta_2 \), and let the modulus \( |f(re^{i\theta})| \) have radial limit 1 for almost all \( e^{i\theta} \) on \( A \). Then a necessary and sufficient condition that \( f(z) \) may be continued analytically across the arc \( A \) by means of the reflection principle \( f(z) = 1/f(1/z) \) is that \( f(z) \) admit neither 0 nor \( \infty \) as an asymptotic value on \( A \).

We proceed now to the principal result of this paper.

**Theorem 2.** Let \( f(z) \) be analytic in \( |z| < 1 \) and of class (LP) on an arc \( \alpha < \theta < \beta \) of \( |z| = 1 \). Let \( A \) be an arbitrary sub-arc of \( (\alpha, \beta) \) and \( \xi \) an arbitrary complex number (including \( \infty \)). If there is a neighborhood

\(^1\) The method of proof of the previous paragraph shows also that \( \infty \) is an asymptotic value of \( F(z) \), but the presence of the factor \( B_1(z) \) precludes an immediate inference that \( \infty \) is an asymptotic value of \( \phi(z) \), and, consequently, of \( f(z) \).
V(e^{i\theta_A}) \cap K of the midpoint e^{i\theta_A} of A in which the \( z \)-points of \( f(z) \) satisfy (1), then there exists a point \( e^{i\theta_0} \) on that part of \( A \) which bounds \( V \cap K \), and a Jordan arc \( \mathcal{L} \) of \( \| z \| < 1 \) terminating at \( e^{i\theta_0} \) such that \( f(z) \rightarrow \xi \) as \( z \rightarrow e^{i\theta_0} \) along \( \mathcal{L} \).

Let us suppose that there exists an arc \( A (\theta_1 < \theta < \theta_2) \), with midpoint \( e^{i\theta_A} \) and contained in \((\alpha, \beta)\), and a complex number \( \xi \) satisfying (1) in some neighborhood \( V(e^{i\theta_A}) \cap K \) which \( f(z) \) does not admit as an asymptotic value on that subarc \( B \) of \( A \) which bounds \( V(e^{i\theta_A}) \cap K \). The case \( \xi = 0 \) being trivial, we may suppose that \( \xi \) is not 0, and, for the moment, not \( \infty \). Since \( f(z) \) is of class \((LP)\) on \( A \), the function \( \phi(z) = \xi^{-1}[e^{i\theta_A} - f(z)] \) is analytic in \( \| z \| < 1 \) and has radial limit 1 for almost all \( e^{i\theta_A} \) on \( A \). Since the \( z \)-points of \( f(z) \) satisfy (1) in \( V(e^{i\theta_A}) \cap K \), we may write, as before, \( \phi(z) = B_1(z) F(z) \), where \( B_1(z) \) is a Blaschke product extended over the zeros of \( \phi(z) \) in \( V(e^{i\theta_A}) \cap K \), and where \( F(z) \) is analytic without zeros in \( V(e^{i\theta_A}) \cap K \). The function \( F(z) \) must possess radial limit values of modulus 1 for almost all \( e^{i\theta_A} \) on \( B \). It cannot happen that \( F(z) \) is the quotient of two bounded functions \( V(e^{i\theta_A}) \cap K \); for otherwise the Riesz-Nevanlinna theorem would imply that \( \phi(z) \), and consequently \( f(z) \), is identically constant. Furthermore, it is clear that no point of \( B \) can be a regular point of \( F(z) \). It follows from Lemma 2 that the set of singularities of \( F(z) \) on \( B \) (namely, all points of \( B \)) is the closure of the set of points \( e^{i\theta_A} \) on \( B \) which are the terminal points of Jordan arcs along which either \( F(z) \rightarrow 0 \) or \( F(z) \rightarrow \infty \). A simple modification of a result of Carathéodory [1, pp. 266–267] and the author [3, p. 251] shows that, unless \( \| F(z) \| > 1 \) in \( V(e^{i\theta_A}) \cap K \), \( F(z) \) must admit zero as an asymptotic value on \( B \). Now if \( \| F(z) \| > 1 \) in \( V(e^{i\theta_A}) \cap K \), then \( \phi(z) \) is the quotient of two bounded functions in that region and, according to the corollary of the Riesz-Nevanlinna theorem, must be identically constant. Since \( F(z) \) must admit 0 as an asymptotic value on \( B \), the boundedness of \( B_1(z) \) implies that 0 is an asymptotic value of \( \phi(z) \) on \( B \), so that \( \xi \) is an asymptotic value of \( f(z) \) on \( B \). This contradiction proves Theorem 2 for the case that \( \| \xi \| < \infty \). For the case that \( \xi = \infty \), we apply Lemma 2 directly to the function \( g(z) = e^{i\xi} \), thus completing the proof of Theorem 2.

References


UNIVERSITY OF MICHIGAN

MULTIPLICATIVE GROUPS OF ANALYTIC FUNCTIONS

WALTER RUDIN

Let $D$ be a proper subdomain of the Riemann sphere, and let $M(D)$ be the multiplicative group of all regular single-valued analytic functions on $D$ which have no zeros in $D$. It is known [1] that the algebraic structure of the ring $R(D)$ of all regular single-valued analytic functions on $D$ determines (and is determined by) the conformal type of $D$. In this paper we ask the question: what information about $D$ does the algebraic structure of $M(D)$ give, and, conversely, which properties of $D$ determine the algebraic structure of $M(D)$? The answer is, briefly, that $M(D_1)$ and $M(D_2)$ are isomorphic if and only if $D_1$ and $D_2$ have the same connectivity.

Here the connectivity of $D$ is $k$ if the complement of $D$ has $k$ components, and is $\infty$ if the complement of $D$ has infinitely many (countable or power of the continuum) components. The structure of $M(D)$ is described in more detail in the theorem below.

If we associate with each $f \in M(D)$ the function $g = f/|f|$ we obtain a subgroup (isomorphic to $M(D)$) of the multiplicative group $C(D)$ of all continuous functions from $D$ into the unit circumference. Such functions have been studied in great detail by Eilenberg [2]. It is worth noting that our theorem is valid if we replace $M(D)$ by $C(D)$, and that the proof is essentially the same; but it seems more interesting to stay within the smaller group.

Before stating the theorem, it is convenient to define two subgroups of $M(D)$.

1. Fix a point $z_0 \in D$ and let $G(D)$ be the set of all $f \in M(D)$ such that $f(z_0) = 1$. Then $M(D)$ is the direct product of $G(D)$ and the multiplicative group of the nonzero complex numbers, and $G(D)$ ...