ON THE RADIAL LIMITS OF ANALYTIC FUNCTIONS

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1. Examples have been given [5, p. 185] of functions \( f(z) \), analytic in the unit-circle \( K: |z| < 1 \), and not identically constant, for which the radial limit \( f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}) \) is zero for all \( e^{i\theta} \) on \( |z| = 1 \) except for a set of linear measure zero. In view of the Riesz-Nevanlinna theorem [6, p. 197], such functions cannot be bounded, or even of bounded characteristic, in \( |z| < 1 \). Functions of this sort appear again whenever we have an analytic function \( f(z) \) whose radial limits coincide almost everywhere with the radial limits of a bounded analytic function \( g(z) \), for the difference \( F(z) = f(z) - g(z) \) has a radial limit zero almost everywhere on \( |z| = 1 \). The Riesz-Nevanlinna theorem shows that, if \( f(z) \) is bounded, or of bounded characteristic, and if the radial limit values of \( f(z) \) coincide almost everywhere on an arc of \( |z| = 1 \) with the radial limit values of \( g(z) \), then \( F(z) \) must be identically zero in \( |z| < 1 \). The object of this note is to discuss certain aspects of the behavior of nonconstant analytic functions whose radial limits vanish almost everywhere on an arc \( A (\theta_1 < \theta < \theta_2) \) of \( |z| = 1 \). One result of such a study (which the author plans as a sequel to this note) will be to give some idea of the way in which a function \( f(z) \), whose radial limits coincide almost everywhere with the radial limits of a function \( g(z) \) of bounded characteristic, can differ from \( g(z) \).

We shall say that a nonconstant function \( f(z) \), analytic in \( |z| < 1 \), is of class \( (LP) \) on an arc \( A \) of \( |z| = 1 \), if \( \lim_{r \to 1} f(re^{i\theta}) = f(e^{i\theta}) = 0 \) for almost all \( e^{i\theta} \) belonging to the arc \( A \). If the arc \( A \) is the whole circumference \( |z| = 1 \), we shall say simply that the function \( f(z) \) is of class \( (LP) \).

One property of functions which are of class \( (LP) \) on an arc \( A \) is immediate: the cluster set of \( f(z) \) at each point \( e^{i\theta_0} \) of \( A \) (i.e., the set of all values \( \alpha \) with the property that there exists a sequence \( \{z_n\}, |z_n| < 1, \lim_{n \to \infty} z_n = e^{i\theta_0} \), such that \( \lim_{n \to \infty} f(z_n) = \alpha \) is the whole plane. For, if there is a point \( e^{i\theta_0} \) on \( A \) and a complex number \( \alpha \) which does not belong to the cluster set of \( f(z) \) at \( e^{i\theta_0} \), then there is a circular neighborhood \( V(e^{i\theta_0}) \) of \( e^{i\theta_0} \) such that, in \( V(e^{i\theta_0}) \cap K \), the function \( g(z) = [f(z) - \alpha]^{-1} \) is analytic and bounded. Since the function \( g(z) \) has the constant limit \(-1/\alpha\) along almost all normal segments drawn to that arc of \( |z| = 1 \) which bounds \( V \cap K \), it follows from a simple corollary of the Riesz-Nevanlinna theorem that \( g(z) \), and hence \( f(z) \),

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must be identically constant in \( V \cap K \) and, a fortiori, in \(|z| < 1\). This property, i.e., that the cluster set is the whole plane, sometimes called the Weierstrass property, suggests that we investigate the values \( \xi \) which \( f(z) \) admits as asymptotic values, i.e., the values to which \( f(z) \) tends as \( z \) approaches a point \( P \) of \(|z| = 1\) along a curve terminating at \( P \). We shall show (Theorem 1) that every complex value \( \xi \) (including \( \infty \)) is an asymptotic value of a function \( f(z) \) of class \((LP)\) provided that the \( \xi \)-points, i.e., the points \( z_k \) for which \( f(z_k) = \xi \), satisfy the condition

\[
\sum_{k=1}^{\infty} (1 - |z_k|) < \infty.
\]

In Theorem 2 we show that a function of class \((LP)\) on an arc \( A \) admits every complex number \( \xi \) as an asymptotic value in every neighborhood of every point \( e^{i\theta} \) of \( A \) if the \( \xi \)-points in some neighborhood \( V(e^{i\theta}) \cap K \) of \( e^{i\theta} \) satisfy (1). Theorem 2 then contains Theorem 1, but the proof of Theorem 1 is considerably simpler, and we give a separate proof.

**Lemma 1.** Let \( f(z) \) be analytic and different from 0 in \(|z| < 1\), and let the modulus \(|f(re^{i\theta})|\) have radial limit 1 for almost all \( e^{i\theta} \) on \(|z| = 1\). Then unless \( f(z) \) is identically constant in \(|z| < 1\), there exists a Jordan arc \( \mathcal{L} \), lying in \(|z| < 1\) and terminating at a point \( e^{i\theta_0} \) of \(|z| = 1\), such that, as \( z \to e^{i\theta_0} \) along \( \mathcal{L} \), either \( f(z) \to 0 \) or \( f(z) \to \infty \). If there exists no path along which \( f(z) \to 0 \), then \(|f(z)| > 1\) in \(|z| < 1\).

Lemma 1 is equivalent to Theorems 5 and 6 of [3], and its proof is omitted here. For brevity, we shall say that a function which is analytic in \(|z| < 1\) and whose modulus \(|f(re^{i\theta})|\) has radial limit 1 for almost all \( e^{i\theta} \) on \(|z| = 1\) will be called of class \((U)\) in \(|z| < 1\).

**Theorem 1.** If \( f(z) \) is analytic in \(|z| < 1\) and of class \((LP)\), then every complex number (including \( \infty \)) which satisfies (1) is an asymptotic value of \( f(z) \).

Assume that a finite \( \xi \) satisfying (1) is not an asymptotic value of \( f(z) \); clearly, we need not consider the case that \( \xi = 0 \). Since \( f(z) \) is of class \((LP)\), the function \( \phi(z) = \xi^{-1}[\xi - f(z)] \) has radial limit 1 almost everywhere and is then of class \((U)\) in \(|z| < 1\). Because the \( \xi \)-points of \( f(z) \) satisfy (1), we may write \( \phi(z) = B_\xi(z)F(z) \), where \( B_\xi(z) \) is a Blaschke product extended over the zeros of \( \phi(z) \). It is well known [7, p. 94] that the radial limits of a Blaschke product exist and have modulus 1 almost everywhere on \(|z| = 1\). From this it follows that \( F(z) \) is of class \((U)\) without zeros in \(|z| < 1\). It is then a conse-

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sequence of Lemma 1 that, unless $F(z)$ is identically constant, $F(z)$ must admit either 0 or $\infty$ as an asymptotic value. We remark first that $F(z)$ cannot be constant; for if $\phi(z)$ reduces to a Blaschke product whose radial limit is 1 almost everywhere, the Riesz-Nevanlinna theorem shows that $\phi(z)$, and consequently $f(z)$, is constant. We assert next that 0 must be an asymptotic value of $F(z)$; otherwise $|F(z)| > 1$ in $|z| < 1$, so that $\phi(z)$ could be expressed as the quotient of two bounded functions in $|z| < 1$, i.e., $\phi(z)$ would be of bounded characteristic in $|z| < 1$. Again, by the Riesz-Nevanlinna theorem, $\phi(z)$ would be constant in $|z| < 1$. Since zero must now be an asymptotic value of $F(z)$, and since $B_1(z)$ is bounded, $\phi(z)$ must admit zero as an asymptotic value, so that $\xi$ is an asymptotic value of $f(z)$.

To show that $f(z)$ admits $\infty$ as an asymptotic value, we remark that the function $g(z) = e^{\phi(z)}$ is of class $(U)$ without zeros in $|z| < 1$. Applying Lemma 1 to $g(z)$, we see that, since $g(z)$ is not constant, $g(z)$ admits either 0 or $\infty$ as an asymptotic value, so that there exists at least one path $L$ terminating at some point $e^{i\theta_0}$ of $|z| = 1$ along which $f(z) \to \infty$. Thus Theorem 1 is proved.

We remark that Theorem 1 is related to a recent result of Cartwright and Collingwood [2, p. 112], the added hypothesis that $f(z)$ be of class $(LP)$ in $|z| < 1$ allowing us to obtain a stronger conclusion to part of Theorem 9 of their paper.

2. In order to determine how frequently a function of class $(LP)$ admits as an asymptotic value a complex number $\xi$ satisfying (1), it will be necessary to use a form of the Schwarz reflection principle developed recently in [4]. We summarize this principle as a lemma.

**Lemma 2.** Let $f(z)$ be meromorphic in $|z| < 1$ and let $A$ be the arc $0 \leq \theta_1 < \theta < \theta_2 < 2\pi$. Let there exist an $\varepsilon > 0$ such that $f(z)$ has no zeros or poles in the region $0 < 1 - |z| < \varepsilon$, $\theta_1 < \arg z < \theta_2$, and let the modulus $|f(re^{i\theta})|$ have radial limit 1 for almost all $e^{i\theta}$ on $A$. Then a necessary and sufficient condition that $f(z)$ may be continued analytically across the arc $A$ by means of the reflection principle $f(\overline{z}) = 1/f(1/z)$ is that $f(z)$ admit neither 0 nor $\infty$ as an asymptotic value on $A$.

We proceed now to the principal result of this paper.

**Theorem 2.** Let $f(z)$ be analytic in $|z| < 1$ and of class $(LP)$ on an arc $0 < \theta < \beta$ of $|z| = 1$. Let $A$ be an arbitrary sub-arc of $(\alpha, \beta)$ and $\xi$ an arbitrary complex number (including $\infty$). If there is a neighborhood 

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1 The method of proof of the previous paragraph shows also that $\infty$ is an asymptotic value of $F(z)$, but the presence of the factor $B_1(z)$ precludes an immediate inference that $\infty$ is an asymptotic value of $\phi(z)$, and, consequently, of $f(z)$.
$V(e^{i\alpha}A) \cap K$ of the midpoint $e^{i\alpha}$ of $A$ in which the $\xi$-points of $f(z)$ satisfy (1), then there exists a point $e^{i\theta_0}$ on that part of $A$ which bounds $V \cap K$, and a Jordan arc $\mathcal{L}$ of $|z| < 1$ terminating at $e^{i\theta_0}$ such that $f(z) \to \xi$ as $z \to e^{i\theta_0}$ along $\mathcal{L}$.

Let us suppose that there exists an arc $A (\theta_1 < \theta < \theta_3)$, with midpoint $e^{i\alpha}$ and contained in $(\alpha, \beta)$, and a complex number $\xi$ satisfying (1) in some neighborhood $V(e^{i\alpha}A) \cap K$ which $f(z)$ does not admit as an asymptotic value on that subarc $B$ of $A$ which bounds $V(e^{i\alpha}A) \cap K$. The case $\xi = 0$ being trivial, we may suppose that $\xi$ is not 0, and, for the moment, not $\infty$. Since $f(z)$ is of class (LP) on $A$, the function $\phi(z) = \xi^{-1} [z - f(z)]$ is analytic in $|z| < 1$ and has radial limit 1 for almost all $e^{i\theta}$ on $A$. Since the $\xi$-points of $f(z)$ satisfy (1) in $V(e^{i\alpha}A) \cap K$, we may write, as before, $\phi(z) = B_1(z) F(z)$, where $B_1(z)$ is a Blaschke product extended over the zeros of $\phi(z)$ in $V(e^{i\alpha}A) \cap K$, and where $F(z)$ is analytic without zeros in $V(e^{i\alpha}A) \cap K$. The function $F(z)$ must possess radial limit values of modulus 1 for almost all $e^{i\theta}$ on $B$. It cannot happen that $F(z)$ is the quotient of two bounded functions $V(e^{i\alpha}A) \cap K$; for otherwise the Riesz-Nevanlinna theorem would imply that $\phi(z)$, and consequently $f(z)$, is identically constant. Furthermore, it is clear that no point of $B$ can be a regular point of $F(z)$. It follows from Lemma 2 that the set of singularities of $F(z)$ on $B$ (namely, all points of $B$) is the closure of the set of points $e^{i\theta}$ on $B$ which are the terminal points of Jordan arcs along which either $F(z) \to 0$ or $F(z) \to \infty$. A simple modification of a result of Carathéodory [1, pp. 266–267] and the author [3, p. 251] shows that, unless $|F(z)| > 1$ in $V(e^{i\alpha}A) \cap K$, $F(z)$ must admit zero as an asymptotic value on $B$. Now if $|F(z)| > 1$ in $V(e^{i\alpha}A) \cap K$, then $\phi(z)$ is the quotient of two bounded functions in that region and, according to the corollary of the Riesz-Nevanlinna theorem, must be identically constant. Since $F(z)$ must admit 0 as an asymptotic value on $B$, the boundedness of $B_1(z)$ implies that 0 is an asymptotic value of $\phi(z)$ on $B$, so that $\xi$ is an asymptotic value of $f(z)$ on $B$. This contradiction proves Theorem 2 for the case that $|\xi| < \infty$. For the case that $\xi = \infty$, we apply Lemma 2 directly to the function $g(z) = e^{i\theta}$, thus completing the proof of Theorem 2.

References


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MULTIPLICATIVE GROUPS OF ANALYTIC FUNCTIONS

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Let $D$ be a proper subdomain of the Riemann sphere, and let $M(D)$ be the multiplicative group of all regular single-valued analytic functions on $D$ which have no zeros in $D$. It is known [1] that the algebraic structure of the ring $R(D)$ of all regular single-valued analytic functions on $D$ determines (and is determined by) the conformal type of $D$. In this paper we ask the question: what information about $D$ does the algebraic structure of $M(D)$ give, and, conversely, which properties of $D$ determine the algebraic structure of $M(D)$? The answer is, briefly, that $M(D_1)$ and $M(D_2)$ are isomorphic if and only if $D_1$ and $D_2$ have the same connectivity.

Here the connectivity of $D$ is $k$ if the complement of $D$ has $k$ components, and is $\infty$ if the complement of $D$ has infinitely many (countable or power of the continuum) components. The structure of $M(D)$ is described in more detail in the theorem below.

If we associate with each $f \in M(D)$ the function $g = f/|f|$ we obtain a subgroup (isomorphic to $M(D)$) of the multiplicative group $C(D)$ of all continuous functions from $D$ into the unit circumference. Such functions have been studied in great detail by Eilenberg [2]. It is worth noting that our theorem is valid if we replace $M(D)$ by $C(D)$, and that the proof is essentially the same; but it seems more interesting to stay within the smaller group.

Before stating the theorem, it is convenient to define two subgroups of $M(D)$.

(1) Fix a point $z_0 \in D$ and let $G(D)$ be the set of all $f \in M(D)$ such that $f(z_0) = 1$. Then $M(D)$ is the direct product of $G(D)$ and the multiplicative group of the nonzero complex numbers, and $G(D)$