SYMMETRIC POLYNOMIALS WITH NON-NEGATIVE COEFFICIENTS

L. KUIPERS AND B. MEULENBEKEL

1. Introduction. Brunn [1] proved a theorem on a determinant (an alternant) the elements of which are elementary symmetric functions of positive variables. This theorem reflected by us in a sharper form and applied to polynomials (Theorem 1) is the basis of further investigation. A special case is Theorem 2, important applications of which are Theorems 3 and 4. In §2 we prove these results and in §3 we give some examples. In §4 the foregoing is applied to absolutely monotonic functions; the result is Theorem 5, a generalization of a theorem of Rosenbloom [2]. In §5 an extension of Theorem 3 is deduced (Theorem 6) by considering a function of two variables.

2. Let $S_j$ be the elementary symmetric function of $n$ variables $x_1, x_2, \ldots, x_n$, defined by

$$S_j = \sum x_1x_2 \cdots x_j$$

for $j = 1, 2, \ldots, n$;

$$S_0 = 1; S_j = 0$$

for $j = -1, -2, \ldots$ and $j > n$.

Theorem 1. The determinant $(S_{ki})$, $i=1, 2, \ldots, q; j=1, 2, \ldots, q$, with

$$k_{i,m} - k_{i,m+1} = k(m) > 0,$$

$$k_{m+1,i} - k_{m,i} = k^*(m) > 0,$$

$i = 1, 2, \ldots, q; m = 1, 2, \ldots, q - 1,$

can be written as a symmetric polynomial in $x_1, x_2, \ldots, x_n$ with non-negative coefficients.

Proof. Obviously the determinant in question is a symmetric polynomial in the considered variables.

For $n=1$ the determinant equals zero or unity or a power of $x_1$. Now applying induction we assume the assertion to be true for $n-1$ and prove the truth for the case $n$. Therefore we put

$$(2) \quad S_k = S_k' + x_nS_{k-1}'$$

where $S_k'$ differs from $S_k$ in referring to the variables with $x_n$ left out. Substituting (2) in the determinant we can expand this in increasing powers of $x_n$, hence

Received by the editors May 12, 1952 and, in revised form, April 4, 1954.

88
SYMMETRIC POLYNOMIALS 89

\[(S_{kij}) = \sum_{k=0}^{q} A_k x_n^k,\]

where \(A_k\) is a sum of determinants the elements of which are elementary symmetric functions of \(x_1, x_2, \cdots, x_{n-1}\). The element indices satisfy (1), hence from our assumption it follows that each \(A_k\) is a polynomial in \(x_1, x_2, \cdots, x_{n-1}\) with non-negative coefficients. Because of (3) our assertion is proved.

An immediate consequence of Theorem 1 is

**Theorem 2.** Let \(\sigma_j = (-1)^j S_j\). Then the determinant

\[\left(\sigma_{m+i-k_j}\right) (i = 0, 1, \cdots, q; j = 0, 1, \cdots, q; 0 = k_0 < k_1 < \cdots < k_d)\]

multiplied by the factor \((-1)^M\), where

\[M = m + (m - k_1) + (m - k_2) + \cdots\]

\[+ (m - k_d) + 1 + 2 + \cdots + q,\]

is expressible as a symmetric polynomial in \(x_1, x_2, \cdots, x_n\) with non-negative coefficients.

Now we prove the following

**Theorem 3.** Let

\[f_{h+1}(x) = a_{h0} + a_{h1}x + \cdots + a_{h,n+p}x^{n+p} \quad (h = 0, 1, \cdots, n - 1),\]

where \(n \geq 2\) and \(p \geq 0\), be \(n\) polynomials with real coefficients such that all determinants \(D\) of the \(n\)th order, taken from the matrix \(|a_{ij}|, i = 0, 1, \cdots, n - 1; j = 0, 1, \cdots, n + p,\) are non-negative.

If for \(n\) variables \(x_1, x_2, \cdots, x_n\), with \(x_i \neq x_j\) for \(i \neq j\), we put \(V = V(x) = \) the determinant \((x_i^j), i = 1, 2, \cdots, n; j = 0, 1, \cdots, n - 1,\) then the expression

\[(4) \quad (f_i(x_i))/V.\]

can be written as a symmetric polynomial in \(x_1, x_2, \cdots, x_n\) with non-negative coefficients.

**Proof.** From a theorem of Garbieri [3] it follows that (4) is equal to the determinant of \((n+p+1)\)th order

\[(B_{ij}), B_{ij} = a_{ij} \quad (i = 0, 1, \cdots, n - 1; j = 0, 1, \cdots, n + p),\]

\[B_{ij} = \sigma_{i-j} \quad (i = n, n + 1, \cdots, n + p; j = 0, 1, \cdots, n + p),\]

where \(\sigma_j\) is defined in Theorem 2. By expanding (5) in terms of the \((p+1)\)-line minors of the last \(p+1\) rows (let \(i\) denote the rows) we see that \((B_{ij})\) is the sum of a number of expressions each of which is
a product of a determinant \( D \), the corresponding determinant 
\[
(\sigma_{n+i-k_j}), \quad i=1, 2, \ldots, p+1; \quad j=1, 2, \ldots, p+1; \quad 1 \leq k_1 < k_2 < \cdots < k_{p+1} \leq n+p+1,
\]
and the factor \((-1)^S\) where 
\[
S = (n+1)+(n+2)+\cdots+(n+p+1)+k_1+k_2+\cdots+k_{p+1}.
\]
Now the exponent \( M \) (see Theorem 2) related to this last determinant is equal to 
\[
(n+1-k_1)+\cdots+(n+1-k_{p+1})+1+2+\cdots+p,
\]
so that \( M \equiv S \mod 2 \). From this conclusion and Theorem 2 the assertion follows.

A special case of Theorem 3 (put \( f_h(x) = x^{k_h}, h=1, 2, \ldots, n \)) is

**Theorem 4** (P. C. Rosenbloom [2, p. 459]). If \( k_1, k_2, \ldots, k_n \)
are integers with \( 0 \leq k_1 < k_2 < \cdots < k_n \), then

\[
(x_i^{k_j})/V; \quad i, j = 1, 2, \ldots, n,
\]
is a symmetric polynomial in \( x_1, x_2, \ldots, x_n \) with non-negative coefficients.

**Remark.** The last result can also be obtained by application of a theorem of H. Naegelsbach [4].

Acting in this way we find for the expression (6) the determinant

\[
\begin{vmatrix}
S_n & S_{n-1} \cdot S_n - k_1 + 1 & S_{n-1} \cdot S_n - k_2 + 1 & \cdots & S_{n-1} \cdot S_n - k_n + 1 \\
0 & S_n \cdot S_{n-k_1} & S_n \cdot S_{n-k_2} & \cdots & S_n \cdot S_{n-k_n} \\
& & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & S_1
\end{vmatrix}
\]
whose elements satisfy the conditions of Theorem 1.

**3. Examples.** 1. If for \( h=1, 2, \ldots, n; p \geq 0 \),

\[
f_h(x) = \sum_{i=0}^{n+p} a_h^{i+1-1} x^i
\]
with \( 0 < a_1 < \cdots < a_n \),

then \((f_i(x_j))/V; i, j=1, 2, \ldots, n\), is a symmetric polynomial in
\( x_1, x_2, \ldots, x_n \) with non-negative coefficients.

This follows from the fact that the determinants of the \( n \)th order taken from the matrix

\[
| a_i^{+i-2} |, \quad i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, n + p + 1,
\]
divided by the positive number \( V(a) \) are polynomials in the positive \( a_1, a_2, \ldots, a_n \) with non-negative coefficients, as follows from Theorem 4.

2. If the determinants of the nth order taken from the matrix \( |a_{ij}| \),
\( i=1, 2, \ldots, n; j=1, 2, \ldots, n+p \), are positive, and if
where the $k_h$ are integers with $0 \leq k_1 < \cdots < k_n$, then $(A_{ij})/\sqrt{V}$ $(i, j = 1, 2, \ldots, n)$ is a symmetric polynomial with non-negative coefficients.

**Proof.** Putting $f_i(x) = \sum_{q=0}^{n+p} a_{iq} x^q (i = 1, 2, \ldots, n)$, we have

$$A_{ij} = \sum_{h=1}^{n} x_i^{k_h} \sum_{q=0}^{n+p} a_{iq} x^q = \sum_{h=1}^{n} x_i^{k_h} f_i(x),$$

so that

$$(A_{ij}) = (x_i^{k_h})(f_i(x_i)) \quad (i, j = 1, 2, \ldots, n).$$

Application of Theorem 3 completes the proof.

**4. Theorem 5.** If $f(x)$ and $g(x)$ are power series with non-negative coefficients converging in the interval $0 \leq x < a$, then the expression

$$(7) \quad (-1)^{n-1} \det \left| 1 x_i^1 \cdots x_i^{n-3} f(ux) g(vx_1 \cdots x_{i-1} x_{i+1} \cdots x_n) \right| /V,$$

$i = 1, 2, \ldots, n,$

can be written as a power series in the variables $x_1, x_2, \ldots, x_n, u, v$ with non-negative coefficients converging in the range

$$0 \leq x_1, x_2, \ldots, x_n < a; \quad 0 \leq u \leq 1; \quad 0 \leq v \leq \frac{1}{a^{n-1}}.$$

**Proof.** Putting

$$f(x) = \sum_{q=0}^{\infty} a_{q} x^q, \quad g(x) = \sum_{m=0}^{\infty} b_{m} x^m, \quad a_q \geq 0, b_m \geq 0,$$

then (7) can be expressed as

$$(-1)^{n-1} \sum_{q=0}^{\infty} a_{q} b_{m} v^m \left| 1 x_i^1 \cdots x_i^{n-3} f(vx_1 \cdots x_{i-1} x_{i+1} \cdots x_n) \right| /V$$

$$= (-1)^{n-1} \sum_{q=0}^{\infty} \sum_{m=0}^{\infty} a_{q} b_{m} v^m \left| 1 x_i^1 \cdots x_i^{n-3} f(vx_1 \cdots x_{i-1} x_{i+1} \cdots x_n) \right|^m /V$$

$$= (-1)^{n-1} \sum_{q=0}^{\infty} \sum_{m=0}^{\infty} a_{q} b_{m} v^m \left| x_i^m x_i^{m+1} \cdots x_i^{m+n-3} x_i^{m+q} \right| /V$$

$$= \sum_{q=n-2}^{\infty} \sum_{m=0}^{\infty} a_{q} b_{m} v^m \left| x_i^m x_i^{m+1} \cdots x_i^{m+n-3} x_i^{m+q} \right| /V,$$
so that from Theorem 4 the correctness of this theorem follows.

Remark. For \( n = 3 \) this theorem is a result of Rosenbloom about absolutely monotonic functions \([2]\). (The range of \( u \) and \( v \) given there is not quite correct.)

5. Theorem 6. Let \( F(x, y) \) be the function \( \sum_{i=0}^{n+p} \sum_{j=0}^{n+p} a_{ij} x^i y^j \) \((p \geq 0)\), with the property that all determinants \( D \) of the \( n \)th order taken from the matrix \( \begin{vmatrix} a_{ij} \end{vmatrix} \) are non-negative. Let \((x_1, x_2, \cdots, x_n)\) and \((y_1, y_2, \cdots, y_n)\) with \( x_i \neq x_j, y_i \neq y_j \) \((i \neq j)\) be two sets of variables.

Then the expression

\[
(8) \quad \frac{F(x_i, y_j)}{V(x)V(y)}.
\]

is a polynomial, symmetric in \( x_1, x_2, \cdots, x_n \) as well as in \( y_1, y_2, \cdots, y_n \) with non-negative coefficients.

Proof. On account of another theorem of Garbieri \([3]\) the expression (8) is identical with

\[
(9) \quad (-1)^{n-1}(t_{ij}), \quad i, j = 0, 1, \cdots, n + 2p + 1,
\]

where

\[
t_{ij} = a_{ij} \quad \text{(} i, j = 0, 1, \cdots, n + p) = \sigma_{j-i-p-1} \quad \text{(} i = 0, 1, \cdots, n + p, j = n + p + 1, \cdots, n + 2p + 1) = \sigma_{i-j-p-1} \quad \text{(} i = n + p + 1, \cdots, n + 2p + 1, j = 0, 1, \cdots, n + 2p + 1),
\]

where \((-1)^i \sigma_i\) and \((-1)^j \sigma_j\) are the elementary symmetric functions of \( x_1, x_2, \cdots, x_n \) and \( y_1, y_2, \cdots, y_n \) respectively. We develop the determinant in (9) in terms of the \((p+1)\)-line minors of the last \( p + 1 \) rows. The term corresponding with the minor indicated by the column-indices \( k_1, k_2, \cdots, k_{p+1} \), say \( \mathcal{M} \), possesses the sign

\[
(-1)^{(n+p+2)+(n+p+3)+\cdots+(n+2p+2)+k_1+k_2+\cdots+k_{p+1}} = (-1)^{M}.
\]

If in \( \mathcal{M} \) we replace each \( \sigma_j \) by \((-1)^i \sigma_i\), then the new minor \( \mathcal{M}' \) has the sign

\[
(-1)^{(n-k_1+1)+(n-k_2+1)+\cdots+(n-k_{p+1}+1)+1+2+\cdots+p} = (-1)^{N}.
\]

The complementary minor of \( \mathcal{M} \) with elements \( a_{ij} \) and \( \sigma'_k \), say \( \mathcal{M}' \), can be expanded in terms of the \((p+1)\)-line minors of the last \( p + 1 \) columns. The term in the expansion of \( \mathcal{M}' \) corresponding with the minor \( \mathcal{N} \) indicated by the row-indices \( q_1, q_2, \cdots, q_{p+1} \) is provided with the sign

\[
(-1)^{(n+1)+(n+2)+\cdots+(n+p+1)+q_1+q_2+\cdots+q_{p+1}} = (-1)^{P}.
\]
In \( \mathcal{N} \) again we replace the \( \sigma' \) by \( S' \), and the new minor \( \mathcal{N}' \) has the sign
\[
(-1)^{(n-q_1+1)+(n-q_2+1)+\cdots+(n-q_p+1)+1+2+\cdots+p} = (-1)^Q.
\]

Thus the determinant in (9) is the sum of terms each of which is a product of 3 determinants \( \mathcal{M}', \mathcal{R}', D \) and the factor \((-1)^{M+N+P+Q} \). By simple calculation we see that
\[
M + N + P + Q \equiv (\phi + 1)^2 \pmod{2}.
\]
As from our assumption the determinants \( D \) are non-negative, it follows that each term in the development of (9) has the positive sign, on account of \((-1)^{(p+1)+(p+1)^2} = 1 \).

The application of Theorem 1 to each \( \mathcal{M}' \) and \( \mathcal{N}' \) completes the proof.

References


University of Indonesia and
Technische Hogeschool, Delft