A NOTE ON A THEOREM OF ROTH

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In a recent paper Roth [1] proved:

**Theorem 1.** If A and B are $n \times n$ matrices with elements in the field $F$, whose characteristic polynomials are

$$a_0(x^2) - xa_1(x^2) \quad \text{and} \quad b_0(x^2) - xb_1(x^2)$$

respectively, where $a_0(x)$, $a_1(x)$, $b_0(x)$, $b_1(x)$ are elements in the polynomial domain, $F[x]$, of $F$; and if the rank of $A - B$ does not exceed unity; then the characteristic polynomial of $AB$ is

$$(-1)^n [a_0(x)b_0(x) - xa_1(x)b_1(x)].$$

In this note the following extension of this theorem will be proved:

**Theorem 2.** If A and B are as given in Theorem 1 and $k$ is any element of the field $F$, then the characteristic polynomial of $AB + k(A - B)$ is

$$(-1)^n [a_0(x)b_0(x) - xa_1(x)b_1(x) + k[a_0(x)b_1(x) - a_1(x)b_0(x)]].$$

Use the notation of Roth and write $A - B = D = R^T S$ where $R$ and $S$ are row vectors in $n$-space.

Since

$$|Ix - A| = a_0(x^2) - xa_1(x^2) \quad \text{and} \quad |Ix - B| = b_0(x^2) - xb_1(x^2)$$

we have $|Ix + A| = (-1)^n [a_0(x^2) + xa_1(x^2)].$

It then follows that

$$|Ix^2 - AB + x(A - B)| = (-1)^n \left\{ a_0(x^2)b_0(x^2) - x^2a_1(x^2)b_1(x^2) \\
+ x[a_1(x^2)b_0(x^2) - a_0(x^2)b_1(x^2)] \right\}. \quad (1)$$

From Lemma II of Roth's paper, it follows that

$$|Ix^2 - AB + x(A - B)| = |Ix^2 - AB| + xS[\text{adj}(Ix^2 - AB)]R^T. \quad (2)$$

The right members of (1) and (2) are identically equal and each is written as the sum of an even polynomial and an odd polynomial. Equating corresponding parts gives

$$|Ix^2 - AB| = (-1)^n [a_0(x^2)b_0(x^2) - x^2a_1(x^2)b_1(x^2)] \quad \text{and} \quad (3)$$

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\( (4) \ S[\text{adj} (Ix^2 - AB)]R^T = (-1)^n[a_1(x^2)b_0(x^2) - a_0(x^2)b_1(x^2)]. \)

Also from Lemma II (Roth), it follows that

\[
\begin{align*}
|Ix^2 - [AB + k(A - B)]| &= |Ix^2 - AB| \\
&\quad - kS[\text{adj} (Ix^2 - AB)]R^T.
\end{align*}
\]

Replacing \( x^2 \) by \( y \) and substituting from (3) and (4) into (5) gives

\[
\begin{align*}
|Iy - [AB + k(A - B)]| &= (-1)^n\left\{a_0(y)b_0(y) - ya_1(y)b_1(y) + \lambda[a_0(y)b_1(y) - a_1(y)b_0(y)]\right\}.
\end{align*}
\]

This completes the proof of the theorem.

The characteristic polynomials of many other functions of \( A \) and \( B \) may now be written at once. Since \((A - B)^u = (SR^T)^{u-1}(A - B)\) for every positive integer \( u \), the characteristic polynomial of \( AB + c(A - B)^u \) may be obtained from (6) by taking \( k = c(SRT)^{u-1} \). In particular for \( c = 1 \) and \( u = 2 \) this becomes \( AB + (A - B)^2 = A^2 - BA + B^2 \). Thus (6) may be used to write the characteristic polynomial of \( A^2 - BA + B^2 \).

Also Lemma II gives the characteristic polynomial of \( A - B \) to be \( x^n - SRTx^{n-1} \). Hence, the matrix \( R^TS \) is nilpotent if and only if the vectors \( R \) and \( S \) are orthogonal.

**Reference**


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