ment a which suffice. Let \( F = B(t) \), \( B \) any field of characteristic two and \( t \) transcendental over \( B \). If \( f(t) \) denotes an arbitrary element of \( B(t) \), then define \( \alpha \) by \( f(t)\alpha = f(1/t) \), and let \( a = t + 1/t \).

**Bibliography**


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**ON THE CHARACTERISTIC FUNCTION OF A MATRIX PRODUCT**

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In a recent note [1], Roth has proved this result.

**Theorem 1.** Let \( A \) and \( B \) be \( n \times n \) matrices, with elements in a field \( F \), and let

\[
|xI - A| = a_0(x^2) - xa_1(x^2), \quad |xI - B| = b_0(x^2) - xb_1(x^2),
\]

where \( a_0, a_1, b_0, \) and \( b_1 \) are elements in the polynomial ring \( F[x] \). If the rank of \( A - B \) is not greater than unity, then

\[
|xI - AB| = (-)^n[a_0(x)b_0(x) - xa_1(x)b_1(x)].
\]

In his proof, which is essentially a verification, Roth derives some interesting but unnecessary information. Here I present a proof which is shorter, direct, and leads naturally to a more general result involving three matrices.

The essential step in my proof is the observation that if \( A \) is a nonsingular matrix and \( M \) is a matrix of rank 1, then

\[
|A + M| = |A| + \sum \Delta_i
\]

where \( \Delta_i \) is a sum of \( n \) determinants, each consisting of \( n - 1 \) columns of \( A \) and one column of \( M \). This follows from the fact that, \( M \) being of rank 1, any two columns of \( M \) are linearly dependent.

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For the case at hand we have
\[
\begin{vmatrix} xI - A \end{vmatrix} \begin{vmatrix} xI + B \end{vmatrix} = \begin{vmatrix} (xI - A)(xI + B) \end{vmatrix} = \begin{vmatrix} x^2I - AB - x(A - B) \end{vmatrix}
\]
and this determinant is equal to \( |x^2I - AB| \) if \( A - B \) has zero rank, while if \( A - B \) has rank 1, we have
\[
\begin{vmatrix} x^2I - AB - x(A - B) \end{vmatrix} = \begin{vmatrix} x^2I - AB \end{vmatrix} - x \sum \Delta_i,
\]
where each determinant \( \Delta_i \) has \( n - 1 \) columns chosen from \( x^2I - AB \) and one column from \( A - B \). It is observed that the terms of \( x \sum \Delta_i \) contain only odd powers of \( x \). Thus, in either case, \( |x^2I - AB| \) is equal to the even part of \( |xI - A| \begin{vmatrix} xI + B \end{vmatrix} \). Now
\[
\begin{vmatrix} xI - A \end{vmatrix} \begin{vmatrix} xI + B \end{vmatrix} = (-)^n[a_0(x^2) - xa_1(x^2)][b_0(x^2) + xb_1(x^2)],
\]
and the even part is \( (-)^n[a_0(x^2)b_0(x^2) - x^2a_1(x^2)b_1(x^2)] \). Hence, writing \( y = x^2 \), we have
\[
\begin{vmatrix} yI - AB \end{vmatrix} = (-)^n[a_0(y)b_0(y) - ya_1(y)b_1(y)],
\]
and this is Roth's result.

Before extending this result we prove the

**Lemma.** If \( H \) and \( K \) are nonzero square matrices, such that \( xH - K \) is of rank 1, for \( x \) indeterminate over the field \( F \), then either

(i) \( H = uh', K = uk' \),

or

(ii) \( H = uh', K = vh' \),

where \( u, v, h, k \) are column vectors. Conversely, if \( H \) and \( K \) satisfy (i) and (ii) then \( xH - K \) is of rank 1.

**Proof.** Since \( xH - K \) is of rank 1 for all \( x \), it follows that \( H \) and \( K \) are each of rank 1 and hence are of the form
\[
H = uh', \quad K = vk',
\]
where \( u, v, h, k \) are column vectors. If we now equate to zero all the two-rowed minors of \( xH - K \), it is easily found that either \( u = v \) or \( h = k \), and this proves the lemma. The converse is obviously true.

From this lemma we proceed to

**Theorem 2.** Let \( A_1, A_2, \) and \( A_3 \) be \( n \times n \) matrices, such that
\[
\begin{vmatrix} xI - A_i \end{vmatrix} = a_{0i}(x^2) + xa_{1i}(x^2) + x^2a_{2i}(x^2) \quad (i = 1, 2, 3)
\]
and write \( H = A_1 + A_2 + A_3 \), \( K = A_1A_2 + A_1A_3 + A_2A_3 \). If \( H \) and \( K \) satisfy the lemma, or if \( H = K = 0 \), then
where $a_{ij} = a_{ij}(x)$.

**Proof.** We have

$$(xI - A_1)(xI - A_2)(xI - A_3) = x^3I - A_1A_2A_3 - x(xH - K).$$

If $H-K = 0$ we have

$$E = \begin{vmatrix} xI - A_1 & | & xI - A_2 & | & xI - A_3 \end{vmatrix} = \begin{vmatrix} x^3I - A_1A_2A_3 \end{vmatrix}.$$  

If $xH - K$ is of rank 1 for all $x$, we have

$$E = \begin{vmatrix} x^3I - A_1A_2A_3 \end{vmatrix} - x \sum \Delta_i,$$

where each determinant $\Delta_i$, since it consists of $n-1$ columns of $x^3I - A_1A_2A_3$ and 1 column of $xH - K$, expands into a polynomial each term of which involves $x$ to the power $3k$ or $3k + 1$ for some integer $k$. Now $x \sum \Delta_i$ is a polynomial, each term of which involves $x$ to a power $3k + 1$ or $3k + 2$. Thus, in either case, $\begin{vmatrix} x^3I - A_1A_2A_3 \end{vmatrix}$ is equal to the sum of the terms of $\begin{vmatrix} xI - A_1 \end{vmatrix} | \begin{vmatrix} xI - A_2 \end{vmatrix} | \begin{vmatrix} xI - A_3 \end{vmatrix}$ which involve powers of $x^3$. If we pick out these terms and replace $x^3$ by $x$ the result follows.

**Reference**


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