ON SYMMETRY IN CONVEX TOPOLOGICAL VECTOR SPACES

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A convex topological vector space $E$ is called symmetric if and only if the topology of the strong bidual of $E$ induces on $E$ its given topology. A barrel in $E$ is a closed, convex, equilibrated, absorbing set (see [1] for the terminology); $E$ is called barreled (French: espace tunnelé) if and only if every barrel is a neighborhood of 0. The properties of being symmetric and of being barreled are but the two extreme examples of a family of properties which we first wish to discuss. Second, we give a new counter-example in the theory of convex topological vector spaces. All spaces considered are assumed Hausdorff.

1. If $E$ and $F$ are convex topological vector spaces, $E'$ the topological dual of $E$, $\Sigma$ a class of bound subsets of $E$ such that $\cup [S | S \in \Sigma] = E$, then $E''$ [respectively $L_2(E, F)$] denotes the vector space $E'\Sigma$ [respectively $L_2(E, F)$], the vector space of all continuous linear transformations from $E$ into $F$ with the (convex) topology of uniform convergence on all members of $\Sigma$. For the important special case where $\Sigma$ is all bound subsets, we write “$b$” for “$\Sigma$”; for that where $\Sigma$ is all one-point subsets of $E$, we write “$s$” for “$\Sigma$.” If $E'$ is a total subspace of the algebraic dual of $E$, among all convex topologies on $E$ yielding $E'$ as dual there is a strongest, denoted by $\tau(E, E')$ [5, Theorem 5]; if a given convex topological vector space $E$ with dual $E'$ has the topology $\tau(E, E')$, it is called relatively strong.

If $E$ is a convex topological vector space, $E'$ its dual, $\Sigma$ a class of bound subsets of $E$ such that $\cup [S | S \in \Sigma] = E$, then $E$ may be canonically identified (algebraically) with a subspace of the vector space (without topology) $(E')'$. Since $E \subset E'$ is equicontinuous if and only if the polar of $L$ in $E$ is a neighborhood of 0, it is easy to see that if $\Lambda$ is a class of bound subsets of $E'$ such that $\cup [L | L \in \Lambda] = E'$, then the topology induced on $E$ by that of $(E')'$ is the given topology of $E$ if and only if $(E')' = (E')_{\Omega}$, where $\Omega$ is the class of all equicontinuous subsets of $E'$. Hence, since every equicontinuous subset of $E'$ is bound in $E'$, the topology of $(E')'$ always induces on $E$ a stronger (i.e., at least as strong) topology than the given topology.

**Definition.** $E$ is $\Sigma$-symmetric if and only if the topology induced...
on $E$ by that of $(E^{'})'$ is the given topology of $E$.

**Theorem 1.** If $\Lambda$ is a class of bound subsets of $E$ such that $\Lambda \supseteq \Sigma$ and if $E$ is $\Sigma$-symmetric, then $E$ is $\Lambda$-symmetric.

**Proof.** As the topology of $E_\Lambda'$ is stronger than that of $E^{'},$ there are fewer bound sets in $E_\Lambda'$ than in $E^{'},$ hence the topology of $(E')'$ is weaker than the topology of $(E^{'})'$, and hence must also induce on $E$ its given topology.

Theorem 1 shows that among all the properties of being $\Sigma$-symmetrical, symmetry (i.e., $b$-symmetry) is the weakest. The strongest such property is $s$-symmetry (i.e., $\Sigma$-symmetry where $\Sigma$ is the class of all one-point subsets of $E$). Theorem 2 shows that this property is precisely the property of being barrelled.

**Theorem 2.** Let $E$ and $F$ be convex topological vector spaces, $F$ of nonzero dimension, $\Sigma$ a class of bound subsets of $E$ such that $\bigcup \{S \mid S \in \Sigma\} = E$. Then the following are equivalent: (1) $E$ is $\Sigma$-symmetric. (2) The polars in $E$ of all bound subsets of $E_\Sigma'$ form a fundamental system of neighborhood of 0 for the topology of $E$. (3) Every bound subset of $E_\Sigma'$ is equicontinuous. (4) Every bound subset of $L_\Sigma(E, F)$ is equicontinuous. (5) Every barrel in $E$ absorbing all members of $\Sigma$ is a neighborhood of 0. (6) $E$ is relatively strong, and every convex bound subset of $E_\Sigma'$ has compact closure in $E_\Sigma'$.

**Proof.** The equivalence of (1), (2), and (3) follows immediately from our discussion above. The equivalence of (3) and (5) follows from the fact that a set is bound in $E_\Sigma'$ if and only if its polar in $E$ is a barrel absorbing all members of $\Sigma$. Proposition 2 of [3] asserts the equivalence of (5) and (6) for the special case of barrelled spaces, and Theorem 1 of [3] asserts that (5) implies (4) for barrelled spaces. In both cases an obvious modification of the proof yields the desired result. It remains to show that (4) implies (3). Let $K_\lambda$ be a one-dimensional subspace of $F$, $K$ the scalar field. $\phi : \lambda K \rightarrow \lambda$ is a topological isomorphism from $K_\lambda$ onto $K$. Let $L = \{u \in L(E, F) \mid u(E) \subseteq K_\lambda\}$ with the topology induced from $L_\Sigma(E, F)$. It is immediate that $\psi : u \rightarrow \phi \circ u$ is a topological isomorphism from $L$ onto $E_\Sigma'$. Let $B$ be bound in $E_\Sigma'$, $V$ a neighborhood of 0 in $K$. Then $V_\lambda = W \cap K_\lambda$ where $W$ is a neighborhood of 0 in $F$. $V_\lambda$ is bound in $L$, hence in $L_\Sigma(E, F)$, and hence is an equicontinuous subset of $L(E, F)$. Therefore there exists a neighborhood $W'$ of 0 in $E$ such that if $u \in \psi^{-1}(B)$ then $u(W') \subseteq W$ and hence, as $u \subseteq L$, $u(W') \subseteq W \cap K_\lambda = V_\lambda$. But then if $v \in B$, $\phi^{-1} \circ v \in \psi^{-1}(B)$ so $\phi^{-1}(v(W')) \subseteq V_\lambda$, i.e., $v(W') \subseteq \phi^{-1}(V_\lambda) = V$. Hence $B$ is equicontinuous.
Corollary 1. A necessary and sufficient condition that $E$ be barrelled is that $E$ be $\Sigma$-symmetric and that every bound subset of $E'$ be bound in $E''$.

"Sequentially complete" (i.e., all Cauchy sequences converge) can replace "complete" in Theorem 2 of [3]; hence

Corollary 2. If every $S \subseteq \Sigma$ is sequentially complete, then $E$ is barrelled if and only if $E$ is $\Sigma$-symmetric.

It is obvious from Theorem 2 that for every theorem about barrelled spaces there is an analogue for $\Sigma$-symmetric spaces. We mention in particular the abstract version of the Banach-Steinhaus theorem, the proof of which for barrelled spaces is found in [2] or in Corollaries 1 and 2 of Theorem 1 of [3].

Theorem 3. Let $E$ be $\Sigma$-symmetric, $\Phi$ a filter on $\mathcal{F}(E, F)$, the vector space of all functions from $E$ into $F$, $u_0 \in \mathcal{F}(E, F)$. Then $u_0 \in \mathcal{L}(E, F)$ and $\Phi$ converges to $u_0$ in the topology of uniform convergence on all precompact subsets of $E$ under any of the following additional assumptions:
(1) $\Phi$ contains a bound subset of $\mathcal{L}_\infty(E, F)$ and $\Phi$ converges pointwise to $u_0$; (2) $F$ is quasi-complete (i.e., every closed, bound subset of $F$ is complete), $\Phi$ contains a bound subset of $\mathcal{L}_\infty(E, F)$, and $\Phi$ converges pointwise to $u_0$ on a total subset of $E$.

2. $E$ is called semi-reflexive if $(E')' = E$ (algebraically). $E$ is called boundedly closed if every bound linear functional on $E$ is continuous. By Theorem 2, a symmetric space is relatively strong. We show the converse is false by giving an example of a semi-reflexive, relatively strong space $F$ whose strong dual $F'$ is a Banach space, but which is neither symmetric nor boundedly closed.

Let $E$ be a nonreflexive Banach space (e.g., $L^1$ of the unit interval). $E$ may be regarded as a total subspace of the algebraic dual of $E'$; we let $F$ be the vector space $E'$ together with the convex topology $\tau(E', E)$. $F$ is thus by definition relatively strong. We show $F' = E$ (algebraically and topologically). Since $E$ is barrelled, the classes of all bound subsets of $E'$, of all bound subsets of $E'$, and of all equicontinuous subsets of $E'$ are identical (Theorem 2); this class is also identical with the class of all bound subsets of $F$, since $E$, the topological dual of $F$, is also the topological dual of $E'$ [4, Theorem 2], and hence $F$ and $E'$ have the same bound subsets [5, Theorem 7]. $V$ is a neighborhood of $0$ in $F'$ if and only if $V$ contains the polar of a bound subset of $F$; $V$ is a neighborhood of $0$ in $E$ if and only if $V$ contains the polar of an equicontinuous subset of $E'$; hence $E$ is
topologically and algebraically identical with $F'$. Thus $(F')' = E' = F$, so $F$ is semi-reflexive. As $E$ is not reflexive, $(E')'$ strictly contains $E$. The topology of $(F')_b' = E_b'$ is thus strictly stronger than that of $F$ since $F$ and $(F')_b'$ have different topological duals; hence $F$ is not symmetric. Also, as the bound subsets of $F$ and of $E_b'$ coincide, every linear functional in $(E')'$ is bound on $F$, but as $F' = E = (E_b')'$, there exist bound linear functionals on $F$ which are not continuous. Hence $F$ is not boundedly closed.

Bibliography

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