ON UNITARY DILATIONS OF CONTRACTIONS

J. J. SCHÄFFER

Using some relatively deep facts about complex functions and spectral measures, B. Sz.-Nagy [2] has recently proved that to every contraction $A$ on a Hilbert space $H$ there corresponds a unitary operator $U$ on a larger Hilbert space $K$ so that $U^n$ is a dilation of $A^n$ for every positive integer $n$. In other words, if $\|A\| \leq 1$, then $K$ and $U$ can be found so that $PU^nP = A^nP$, where $P$ is the projection from $K$ onto $H$. The purpose of this note is to prove the same theorem by directly exhibiting the unitary operator $U$.

Write $K$ for the direct sum of countably many copies of $H$, indexed by the set of all integers. The operator $U$ will be exhibited as a matrix whose entries $U(i, j)$ are operators on $H$ ($i, j = 0, \pm 1, \pm 2, \cdots$). If $P$ denotes the projection from $K$ onto the zeroth coordinate space, then the fact that $PU^nP = A^nP$ will find its matricial expression in the assertion that the $(0, 0)$ entry of $U^n$ is $A^n$. Let $S$ and $T$ be the positive operators on $H$ defined by $S^2 = 1 - AA^*$ and $T^2 = 1 - A^*A$, where 1 denotes the identity operator. The operators $U(i, j)$ are then defined as follows: $U(0, 0) = A$, $U(-1, 1) = -A^*$, $U(0, 1) = S$, $U(i, i + 1) = 1$ when $i < -1$ and when $i > 0$, and $U(i, j) = 0$ in all other cases.

An elementary computation shows that $U$ is unitary once it is proved that $SA = AT$. This was proved by Halmos [1] by noting that $S^2A = AT^2$ and approximating the square root function in the unit interval by polynomials. The fact that $U^n$ is a dilation of $A^n$ is evident since, except for $U(0, 0)$, all the nonzero entries of $U$ are above the main diagonal.

The possibility of such an explicit proof was suggested by P. R. Halmos who also contributed several useful comments.

REFERENCES


UNIVERSITY OF URUGUAY

Received by the editors July 16, 1954.