ON UNITARY DILATIONS OF CONTRACTIONS

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Using some relatively deep facts about complex functions and spectral measures, B. Sz.-Nagy [2] has recently proved that to every contraction A on a Hilbert space H there corresponds a unitary operator U on a larger Hilbert space K so that U^n is a dilation of A^n for every positive integer n. In other words, if $\|A\| \leqq 1$, then K and U can be found so that $PU^nP = A^nP$, where P is the projection from K onto H. The purpose of this note is to prove the same theorem by directly exhibiting the unitary operator U.

Write K for the direct sum of countably many copies of H, indexed by the set of *all* integers. The operator U will be exhibited as a matrix whose entries $U(i, j)$ are operators on H $(i, j = 0, \pm 1, \pm 2, \cdots)$. If P denotes the projection from K onto the zeroth coordinate space, then the fact that $PU^nP = A^nP$ will find its matricial expression in the assertion that the $(0, 0)$ entry of U^n is A^n. Let S and T be the positive operators on H defined by $S^2 = 1 - AA^*$ and $T^2 = 1 - A^*A$, where 1 denotes the identity operator. The operators $U(i, j)$ are then defined as follows: $U(0, 0) = A$, $U(-1, 1) = -A^*$, $U(-1, 0) = T$, $U(0, 1) = S$, $U(i, i+1) = 1$ when $i < -1$ and when $i > 0$, and $U(i, j) = 0$ in all other cases.

An elementary computation shows that U is unitary once it is proved that $SA = AT$. This was proved by Halmos [1] by noting that $S^2A = AT^2$ and approximating the square root function in the unit interval by polynomials. The fact that U^n is a dilation of A^n is evident since, except for $U(0, 0)$, all the nonzero entries of U are above the main diagonal.

The possibility of such an explicit proof was suggested by P. R. Halmos who also contributed several useful comments.

References

1. P. R. Halmos, *Normal dilations and extensions of operators*, Summa Brasiliensis Math. vol. 2 (1950) fasc. 9.

2. B. Sz.-Nagy, *Sur les contractions de l'espace de Hilbert*, Acta Szeged. vol. 15 (1953) pp. 87–92.

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