CONTINUITY OF MULTIPLICATION IN OPERATOR ALGEBRAS

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Parts of Banach algebra theory have been generalized recently [2; 1] to multiplicatively convex topological algebras. An important class of examples of Banach algebras is the class of subalgebras of the algebra of continuous linear transformations on a Banach space. It is natural to ask whether an algebra of continuous linear transformations on a convex vector space, with the topology of bounded convergence (the analogue of the norm topology in an algebra of transformations on a Banach space), is a multiplicatively convex topological algebra. We show that this is in general false, in fact that if the algebra is large enough multiplication is not even continuous unless the convex vector space is normable.

THEOREM. *If X is a Hausdorff convex vector space and multiplication is continuous in the topology of bounded convergence in an algebra $\mathcal{A}$ of continuous linear transformations on X including the subalgebra of transformations with finite-dimensional range, then X is normable.*

PROOF. A fundamental system of neighborhoods of the origin in the topology of bounded convergence in $\mathcal{A}$ is the family of sets $T(A, V)$ of continuous linear transformations u in $\mathcal{A}$ such that $u(A) \subseteq V$, for A a bounded set in X and V a neighborhood of the origin in X. Let $V \neq X$ be an equilibrated ($\lambda V \subseteq V$ for $|\lambda| = 1$) neighborhood of the origin in X and let $A \neq 0$ be a bounded set in X. Then by the continuity of multiplication in $\mathcal{A}$ at the origin there exists a bounded set A' in X and a neighborhood V' of the origin in X such that $[T(A', V')]^2 \subseteq T(A, V)$. V' may be taken to be equilibrated.

We show there exists a scalar $\lambda \neq 0$ such that $\lambda V' \subseteq \bigcup_{u \in T(A', V')} u(A)$. Let $x_0 \neq 0$ be an element of A. By the Hahn-Banach theorem there exists a continuous linear functional l_1 on X with $l_1(x_0) \neq 0$. $l_1(A')$ is bounded. Let $c = \sup_{x \in A'} |l_1(x)|$. Let $\lambda = (l_1(x_0))/c$. Then $\lambda \neq 0$. Given x in V' let ϕ_x be an isomorphism of the scalar field F onto the 1-dimensional subspace $[x]$ generated by x. Let $\lambda_x = \phi_x^{-1}(x)$. Then $\sup_{x \in A'} |(\lambda_x l_1(x))/c| = |\lambda_x|$, so $(\lambda_x l_1(A'))/c \subseteq \phi_x^{-1}(V' \cap [x])$ since $\phi_x^{-1}(V' \cap [x])$ is an equilibrated set in F containing $\phi_x^{-1}(x)$. Let $u_x = (\phi_x \lambda_x l_1)/c$. Then u_x is in $T(A', V')$ since $u_x(A') = (\phi_x \lambda_x l_1(A'))/c \subseteq V' \cap [x] \subseteq V'$. Now $\lambda x = u_x(x_0)$ for all x in V', since $l_1(x_0) = \lambda c$, hence $(\lambda x l_1(x_0))/c = \lambda \lambda_x = \lambda \phi_x^{-1}(x)$ and hence $u_x(x_0) = (\phi_x \lambda_x l_1(x_0))/c = \lambda x$.

Hence $\lambda V' \subseteq \bigcup_{x \in V'} u_x(x_0) \subseteq \bigcup_{u \in T(A', V')} u(A)$.

We show V' is bounded. If V' is not bounded then $\lambda V'$ is also not bounded, and so there exists a continuous linear functional l_2 on X not bounded on $\lambda V'$. Let y be an element of X not in V. Since V is equilibrated $\lambda y \notin V$ if $|\lambda| > 1$. Let ϕ be an isomorphism of F onto $[y]$. Then ϕl_2 is a continuous linear transformation of X into X, hence $\phi l_2(A')$ is bounded, hence absorbed by V', hence there exists $\lambda_1 > 0$ so $\lambda_1 \phi l_2 \in T(A', V')$. Let $u_2 = \lambda_1 \phi l_2$. $u_2 u_x \in [T(A', V')]^2 \subseteq T(A, V)$, but $u_2(\lambda V') \subseteq \bigcup_{x \in V'} u_2 u_x(A)$ and $u_2(\lambda V')$ is an unbounded subset of $[y]$, hence not included in V, a contradiction. Hence V' is bounded.

Since X has a bounded neighborhood V' of the origin, X is a normable space, q.e.d.

It is easy to extend the proof to the case where X is not Hausdorff.

BIBLIOGRAPHY

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