A NOTE ON A THEOREM OF K. G. WOLFSON

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1. In [3] K. G. Wolfson proves the following:

Theorem. A commutative $B^*$-algebra $K$ containing an identity $e$ (with $\|k^*k\| = \|k\|^2$ for all $k \in K$ and $\|e\| = 1$) is isomorphic (in a norm and $*$ preserving manner) to an algebra $B(X)$ of all bounded complex-valued functions on an essentially unique set $X$ if and only if:

1. Every nonzero closed ideal of $K$ contains a minimal ideal;
2. The sum of two annulets is an annulet.

An annulet is here understood to be an ideal $I$ of $K$ with which is associated a subset $G \subseteq K$ such that

$$I = \{k \in K \mid k \cdot g = 0 \text{ for all } g \in G\}.$$

The two characterizing traits therein presented neither are self-evident in $B(X)$ nor touch upon the projections and Gelfand-Nemmarck [1; 2] continuous function representations so commonly used in the analysis of $B^*$-algebras. It is with this in mind that the following characterizations are offered:

Theorem 1. A necessary and sufficient condition that a $B^*$-algebra $C(X)$ of all continuous complex-valued functions on a compact Hausdorff space $X$ be isomorphic (in a norm and $*$ preserving manner) to a $B^*$-algebra $B(X_0)$ of all bounded complex-valued functions on a set $X_0$ (essentially unique) is that the compact space $X$ contain a dense subset $X_0$ of points each of which is an open-closed subset of $X$ and such that each nonempty subset of points in $X_0$ is contained in an open-closed subset of $X$ which includes of $X_0$ just that subset.

Proof. Sufficiency. Assume that the compact Hausdorff space $X$ contains a subset $X_0$ as described above. Viewing $X_0$ simply as a set, form the $B^*$-algebra $B(X_0)$ and consider the map $C(X) \to B(X_0)$ under which $c$ in $C(X)$ becomes $\bar{c}$ in $B(X_0)$ with $\bar{c}(x_0) = c(x_0)$ for all $x_0$ in $X_0$. This map is linear and $*$ preserving. Since $X_0$ is dense in $X$ the map is norm preserving and thus 1-1. Finally this map is onto: thus let $b \in B(X_0)$ be arbitrary and assume that its range as $x_0$ varies over $X_0$ falls in the interior of a right-open square of side $2M$ about the origin in the complex plane. For each positive integer $n$ the de-
composition of this square into $2^n \times 2^n$ equal right-open squares determines a division of $X_0$ into $2^n \times 2^n$ disjoint subsets (some perhaps empty) and thus a decomposition of $X$ into $2^n \times 2^n$ disjoint open-closed subsets (some perhaps the empty set) $X_i$, $i = 1, \ldots, 2^n \times 2^n$. If $\lambda_i$ is the center of the $i$th of the above squares and $\chi(X_i)$ is the characteristic function of the corresponding open-closed subset of $X$, then

$$c_n(x) = \sum_{i=1}^{2^n \times 2^n} \lambda_i \cdot \chi(X_i) [x]$$

is an element of $C(X)$ such that $|\tilde{c}_n(x_0) - b(x_0)| \leq M/n^{1/2}$ for all $x_0$ in $X_0$. Thus $\{\tilde{c}_n\}$ is a Cauchy sequence in $B(X_0)$ and, since norms are preserved, $\{c_n\}$ is a Cauchy sequence in $C(X)$ which converges point-wise (and uniformly) to an element $c \in C(X)$. Finally, for each $x_0$ in $X_0$:

$$c(x_0) = \lim_{n} c_n(x_0) = \lim_{n} \tilde{c}_n(x_0) = b(x_0).$$

**Necessity.** Assume now that $C(X)$ is isomorphic as a $B^*$-algebra to some $B(\overline{X}_0)$. In $B(\overline{X}_0)$ denote by $\tilde{e}_{x_0}$, $\tilde{x}_0 \in \overline{X}_0$, the system of all characteristic functions of a single point $\tilde{x}_0 \in \overline{X}_0$ and by $\tilde{e}_A$, $\tilde{A} \in S(\overline{X}_0)$, the system of all characteristic functions on the various subsets $\tilde{A} \in S(\overline{X}_0)$ of $\overline{X}_0$. Denote by $e_{x_0}$ and $e_A$ the elements of $C(X)$ which correspond under the assumed isomorphism to $\tilde{e}_{x_0}$ and $\tilde{e}_A$ in $B(\overline{X}_0)$. It is clear that $e_{x_0}$ and $e_A^n$ are idempotent, hermitian in $C(X)$ and thus are the characteristic functions of open-closed subsets of $X$. For each $c \in C(X)$ and each $e_{x_0}$, $c \cdot e_{x_0} = \lambda e_{x_0}$ where $\lambda$ is a complex scalar, since the corresponding multiplicative property clearly holds in $B(\overline{X}_0)$. From this it follows, by use of the theorem that for any open set $O$ in compact $X$ and any point $x_0$ in $O$ there is an element $c$ in $C(X)$ with $c(x_0) = 1$ and $c(x) = 0$ for $x \in O$, that $e_{x_0}$ is the characteristic function of a single point $x_0 \in X$. Similarly for each $c \neq 0$ in $C(X)$ there exists at least one $e_{x_0}$ in $C(X)$ such that $c \cdot e_{x_0} \neq 0$ since the corresponding property holds in $B(\overline{X}_0)$. From this it follows by the same theorem that the points $x_0$ determined by the $e_{x_0}$ are dense in $X$. Finally for any collection of the $x_0$ in $X$ the open-closed subset $A$ of $X$ determined by the idempotent, hermitian element $e_A$ of $C(X)$ corresponding to the element $e_A$ of $B(\overline{X}_0)$, where $A$ contains all and only the corresponding $x_0$, itself contains all and only the given $x_0$ since $e_{x_0} \cdot e_A = e_{x_0}$ if and only if $\tilde{e}_{x_0} \cdot \tilde{e}_A = \tilde{e}_{x_0}$.

Since a dense subset in $X$ of points that are open-closed subsets necessarily includes all such points, the essential identity of the sets $\overline{X}_0$, $X_0$ is clear.

2. Let $K$ be an arbitrary commutative $B^*$-algebra with unit $e$, $\|e\| = 1$, and with $\|k \cdot k^*\| = \|k\|^2$ for all $k \in K$. In the collection of all projections (idempotent, hermitian, nonzero elements) of $K$ distin-
guish the collection (possibly empty) $e_{x_0}, x_0 \in X_0$, of projections such that for each $k \in K$ and each $e_{x_0}, x_0 \in X_0$, there exists a complex scalar $\lambda$ such that $k \cdot e_{x_0} = \lambda e_{x_0}$. Such projections may be called minimal projections. Here the indexing set $X_0$ is assumed so chosen that $x_0 \neq y_0$ in $X_0$ implies $e_{x_0} \neq e_{y_0}$ in $K$.

**Theorem 2.** A necessary and sufficient condition that a commutative $B^*$-algebra $K$ with identity $e$ and with $\|k^* \cdot k\| = \|k\|^2$ for all $k$ be isomorphic (in a norm and * preserving manner) to a $B^*$-algebra $B(\overline{X}_0)$ of all complex-valued functions on a set $\overline{X}_0$ is that the subset of all minimal projections $e_{x_0}, x_0 \in X_0$, of $K$ be such that:

1. For each $k \neq 0$ in $K$ there exists at least one minimal projection $e_{x_0}, x_0 \in X_0$, such that $k \cdot e_{x_0} \neq 0$.

2. For each subcollection $A \subseteq X_0$ of minimal projections there exists in $K$ a projection $e_A$ such that $e_A \cdot e_{x_0} = e_{x_0}$ for $x_0 \in A$ and $e_A \cdot e_{x_0} = 0$ for $x_0 \notin A$.

When these conditions are satisfied the indexing set $X_0$ may be identified with the set $\overline{X}_0$.

**Proof.** Necessity. Assume $K$ isomorphic to $B(\overline{X}_0)$. It is then evident that the minimal projections $e_{x_0}, x_0 \in X_0$, of $K$ correspond in a 1-1, onto manner to the characteristic functions in $B(\overline{X}_0)$ of single points $\hat{x}_0$ of $\overline{X}_0$, that the indexing set $X_0$ may be identified with the set $\overline{X}_0$, and that the two conditions given above are satisfied with the $e_A$ of condition (2) being the inverse images in $K$ of the characteristic functions in $B(\overline{X}_0)$ of various subsets $A_0$ of $\overline{X}_0$.

Sufficiency. Assume that the collection (now nonempty) $e_{x_0}, x_0 \in X_0$, of minimal projections in $K$ satisfies the stated conditions. By the Gelfand-Neumark theory [1; 2] $K$ is isomorphic (in a norm and * preserving manner) to an algebra $C(\overline{X})$ of all continuous complex-valued functions on a compact Hausdorff space $\overline{X}$. Let $\overline{X}_0$ denote the set of all points $\hat{x}_0$ which are open-closed sets in $\overline{X}$. Using again the theorem that for each point $\hat{y}$ in an open set $O$ of $\overline{X}$ there exists an element $c$ of $C(\overline{X})$ with $c(\hat{y}) = 1$ and $c(\hat{x}) = 0$ for $\hat{x} \in O$, it is clear that the minimal projections $e_{x_0}, x_0 \in X_0$, of $K$ correspond in a 1-1 onto manner to the collection $e_{x_0}, \hat{x}_0 \in \overline{X}_0$, of all characteristic functions on single point, open-closed sets of $\overline{X}$, that $X_0$ and $\overline{X}_0$ may be identified, that (by condition 1) the points of $\overline{X}_0$ are dense in $\overline{X}$, and that (by condition (2)) each nonempty subset of points in $\overline{X}_0$ is contained in an open-closed subset of $\overline{X}$ which contains all and only those points of $\overline{X}_0$ that are in the given subset. It follows then by Theorem 1 that $C(\overline{X})$ and hence $K$ is isomorphic to a $B(X_0^*)$ wherein the set $X_0^*$ may be identified with $\overline{X}_0$ and thus with $X_0$. 

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A NOTE ON UNSTABLE HOMEOMORPHISMS

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In [1] W. R. Utz introduced the concept of an unstable homeomorphism and raised the question of whether there exists an unstable homeomorphism of a compact continuum onto itself. In this note an example of such an homeomorphism will be given.

Let $C$ denote the complex unit circle and for each $z \in C$, let $g(z) = z^2$. Then $g: C \to C$ determines an inverse limit space $\Sigma_2 = \{(a_0, a_1, a_2, \cdots) \mid \text{for each non-negative integer } i, a_i \in C \text{ and } g(a_{i+1}) = a_i\}$. For $a, b \in \Sigma_2$, the function $\rho(a, b) = \sum_{i=0}^{\infty} \frac{|a_i - b_i|}{2^i}$ is a metric for $\Sigma_2$; $\Sigma_2$ is familiar as the “two-solenoid,” and is a compact, indecomposable continuum. Define $f: \Sigma_2 \to \Sigma_2$ as follows: for each $a = (a_0, a_1, \cdots) \in \Sigma_2$, let $f(a) = [g(a_0), g(a_1), \cdots]$. Then $f(a) = (a_0^2, a_1, \cdots) = (a_0^2, a_0, a_1, \cdots), f^{-1}(a) = (a_1, a_2, a_3, \cdots)$, and $f$ is a homeomorphism of $\Sigma_2$ onto $\Sigma_2$.

To show that $f$ is unstable, suppose that $a = (a_0, a_1, \cdots)$ and $b = (b_0, b_1, \cdots)$ are distinct points of $\Sigma_2$. Consider, as Case 1, that $a_0 \neq b_0$. Let $e^{i\theta} = a_0, e^{i\phi} = b_0$, where $0 \leq \theta, \phi < 2\pi$. Then there exists a non-negative integer $n$ such that the angle between the terminal rays of $2^n\theta$ and $2^n\phi$ is greater than $\pi/2$. Then $\rho[f^n(a), f^n(b)] \geq |a_0^{2^n} - b_0^{2^n}| = |e^{in\theta} - e^{in\phi}| > 1$.

Case 2: for some integer $n > 0$, $a_0 \neq b_n$, but $a_i = b_i$, for $0 \leq i < n$. Then $f^{-n}(a) = (a_n, a_{n+1}, a_{n+2}, \cdots, f^{-n}(b) = (b_n, b_{n+1}, b_{n+2}, \cdots)$, and there-

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2 A homeomorphism $f$ of a compact metric space $X$ onto $X$ is said to be unstable provided there exists a fixed positive number $\delta$, such that if $x$ and $y$ are distinct points of $X$, then there exists an integer $n$, such that $\rho[f^n(x), f^n(y)]$ is greater than $\delta$. 

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