A NOTE ON A THEOREM OF K. G. WOLFSON

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1. In [3] K. G. Wolfson proves the following:

**Theorem.** A commutative B*-algebra \( K \) containing an identity \( e \) (with \( \| k^* \cdot k \| = \| k \|^2 \) for all \( k \in K \) and \( \| e \| = 1 \)) is isomorphic (in a norm and * preserving manner) to an algebra \( B(X) \) of all bounded complex-valued functions on an essentially unique set \( X \) if and only if:

1. Every nonzero closed ideal of \( K \) contains a minimal ideal;
2. The sum of two annulets is an annulet.

An annulet is here understood to be an ideal \( I \) of \( K \) with which is associated a subset \( G \subseteq K \) such that

\[
I = \{ k \in K \mid k \cdot g = 0 \text{ for all } g \in G \}.
\]

The two characterizing traits therein presented neither are self-evident in \( B(X) \) nor touch upon the projections and Gelfand-Neumark [1; 2] continuous function representations so commonly used in the analysis of B*-algebras. It is with this in mind that the following characterizations are offered:

**Theorem 1.** A necessary and sufficient condition that a B*-algebra \( C(X) \) of all continuous complex-valued functions on a compact Hausdorff space \( X \) be isomorphic (in a norm and * preserving manner) to a B*-algebra \( B(X_0) \) of all bounded complex-valued functions on a set \( X_0 \) (essentially unique) is that the compact space \( X \) contain a dense subset \( X_0 \) of points each of which is an open-closed subset of \( X \) and such that each nonempty subset of points in \( X_0 \) is contained in an open-closed subset of \( X \) which includes of \( X_0 \) just that subset.

**Proof.** Sufficiency. Assume that the compact Hausdorff space \( X \) contains a subset \( X_0 \) as described above. Viewing \( X_0 \) simply as a set, form the B*-algebra \( B(X_0) \) and consider the map \( C(X) \rightarrow B(X_0) \) under which \( c \) in \( C(X) \) becomes \( \bar{c} \) in \( B(X_0) \) with \( \bar{c}(x_0) = c(x_0) \) for all \( x_0 \) in \( X_0 \). This map is linear and * preserving. Since \( X_0 \) is dense in \( X \) the map is norm preserving and thus 1-1. Finally this map is onto: thus let \( b \subseteq B(X_0) \) be arbitrary and assume that its range as \( x_0 \) varies over \( X_0 \) falls in the interior of a right-open square of side \( 2M \) about the origin in the complex plane. For each positive integer \( n \) the de-
composition of this square into $2^n \times 2^n$ equal right-open squares determines a division of $X_0$ into $2^n \times 2^n$ disjoint subsets (some perhaps empty) and thus a decomposition of $X$ into $2^n \times 2^n$ disjoint open-closed subsets (some perhaps the empty set) $X_i$, $i = 1, \ldots, 2^n \times 2^n$. If $\lambda_i$ is the center of the $i$th of the above squares and $\chi(X_i)$ is the characteristic function of the corresponding open-closed subset of $X$, then $c_n(x) = \sum_{i=1}^{2^n \times 2^n} \lambda_i \cdot \chi(X_i)[x]$ is an element of $C(X)$ such that $|\tilde{c}_n(x_0) - b(x_0)| \leq M/n^{2^{1/2}}$ for all $x_0 \in X_0$. Thus $\{\tilde{c}_n\}$ is a Cauchy sequence in $B(X_0)$ and, since norms are preserved, $\{c_n\}$ is a Cauchy sequence in $C(X)$ which converges point-wise (and uniformly) to an element $c \in C(X)$. Finally, for each $x_0 \in X_0$:

$$c(x_0) = \lim_{n} c_n(x_0) = \lim_{n} \tilde{c}_n(x_0) = b(x_0).$$

**Necessity.** Assume now that $C(X)$ is isomorphic as a $B^*$-algebra to some $B(\overline{X}_0)$. In $B(\overline{X}_0)$ denote by $\tilde{e}_{x_0}, \tilde{x}_0 \in \overline{X}_0$, the system of all characteristic functions of a single point $\tilde{x}_0 \in \overline{X}_0$ and by $\tilde{\alpha}, \tilde{\alpha} \in S(\overline{X}_0)$, the system of all characteristic functions on the various subsets $\tilde{\alpha} \in S(\overline{X}_0)$ of $\overline{X}_0$. Denote by $e_{\tilde{x}_0}$ and $e_{\tilde{\alpha}}$ the elements of $C(X)$ which correspond under the assumed isomorphism to $\tilde{e}_{x_0}$ and $\tilde{\alpha}$ in $B(\overline{X}_0)$. It is clear that $e_{\tilde{x}_0}$ and $e_{\tilde{\alpha}}$ are idempotent, hermitian in $C(X)$ and thus are the characteristic functions of open-closed subsets of $X$. For each $c \in C(X)$ and each $e_{\tilde{x}_0}$, $c \cdot e_{\tilde{x}_0} = \lambda e_{\tilde{x}_0}$ where $\lambda$ is a complex scalar, since the corresponding multiplicative property clearly holds in $B(\overline{X}_0)$. From this it follows, by use of the theorem that for any open set $O$ in compact $X$ and any point $x_0$ in $O$ there is an element $c$ in $C(X)$ with $c(x_0) = 1$ and $c(x) = 0$ for $x \in O$, that $e_{\tilde{x}_0}$ is the characteristic function of a single point $x_0 \in X$. Similarly for each $c \neq 0$ in $C(X)$ there exists at least one $e_{\tilde{x}_0}$ in $C(X)$ such that $c \cdot e_{\tilde{x}_0} \neq 0$ since the corresponding property holds in $B(\overline{X}_0)$. From this it follows by the same theorem that the points $x_0$ determined by the $e_{\tilde{x}_0}$ are dense in $X$. Finally for any collection of the $x_0$ in $X$ the open-closed subset $A$ of $X$ determined by the idempotent, hermitian element $e_{\tilde{\alpha}}$ of $C(X)$ corresponding to the element $\tilde{e}_{\tilde{\alpha}}$ of $B(\overline{X}_0)$, where $\tilde{\alpha}$ contains all and only the corresponding $\tilde{x}_0$, itself contains all and only the given $x_0$ since $e_{\tilde{\alpha}} \cdot e_{\tilde{\alpha}} = e_{\tilde{x}_0}$ if and only if $\tilde{e}_{\tilde{x}_0} \cdot \tilde{e}_{\tilde{\alpha}} = \tilde{e}_{\tilde{x}_0}$.

Since a dense subset in $X$ of points that are open-closed subsets necessarily includes all such points, the essential identity of the sets $\overline{X}_0$, $X_0$ is clear.

2. Let $K$ be an arbitrary commutative $B^*$-algebra with unit $e$, $\|e\| = 1$, and with $\|k^* \cdot k\| = \|k\|^2$ for all $k \in K$. In the collection of all projections (idempotent, hermitian, nonzero elements) of $K$ distin-
guish the collection (possibly empty) $e_{x_0}, x_0 \in X_0$, of projections such that for each $k \in K$ and each $e_{x_0}, x_0 \in X_0$, there exists a complex scalar $\lambda$ such that $k \cdot e_{x_0} = \lambda e_{x_0}$. Such projections may be called minimal projections. Here the indexing set $X_0$ is assumed so chosen that $x_0 \neq y_0$ in $X_0$ implies $e_{x_0} \neq e_{y_0}$ in $K$.

**Theorem 2.** A necessary and sufficient condition that a commutative $B^*$-algebra $K$ with identity $e$ and with $\|k \cdot k^*\| = \|k\|^2$ for all $k$ be isomorphic (in a norm and $*$ preserving manner) to a $B^*$-algebra $B(\overline{X}_0)$ of all complex-valued functions on a set $\overline{X}_0$ is that the subset of all minimal projections $e_{x_0}, x_0 \in X_0$, of $K$ be such that:

1. For each $k \neq 0$ in $K$ there exists at least one minimal projection $e_{x_0}, x_0 \in X_0$, such that $k \cdot e_{x_0} \neq 0$.

2. For each subcollection $\mathcal{A} \subseteq X_0$ of minimal projections there exists in $K$ a projection $e_{\mathcal{A}}$ such that $e_{\mathcal{A}} \cdot e_{x_0} = e_{x_0}$ for $x_0 \in \mathcal{A}$ and $e_{\mathcal{A}} \cdot e_{x_0} = 0$ for $x_0 \not\in \mathcal{A}$.

When these conditions are satisfied the indexing set $X_0$ may be identified with the set $\overline{X}_0$.

**Proof.** *Necessity.* Assume $K$ isomorphic to $B(\overline{X}_0)$. It is then evident that the minimal projections $e_{x_0}, x_0 \in X_0$, of $K$ correspond in a 1-1, onto manner to the characteristic functions in $B(\overline{X}_0)$ of single points $\bar{x}_0$ of $\overline{X}_0$, that the indexing set $X_0$ may be identified with the set $\overline{X}_0$, and that the two conditions given above are satisfied with the $e_{\mathcal{A}}$ of condition (2) being the inverse images in $K$ of the characteristic functions in $B(\overline{X}_0)$ of various subsets $\mathcal{A}_0$ of $\overline{X}_0$.

* Sufficiency. Assume that the collection (now nonempty) $e_{x_0}, x_0 \in X_0$, of minimal projections in $K$ satisfies the stated conditions. By the Gelfand-Neumark theory [1; 2] $K$ is isomorphic (in a norm and $*$ preserving manner) to an algebra $C(\overline{X})$ of all continuous complex-valued functions on a compact Hausdorff space $\overline{X}$. Let $\overline{X}_0$ denote the set of all points $\bar{x}_0$ which are open-closed sets in $\overline{X}$. Using again the theorem that for each point $\bar{y}$ in an open set $\overline{O}$ of $\overline{X}$ there exists an element $c$ of $C(\overline{X})$ with $c(\bar{y}) = 1$ and $c(\bar{x}) = 0$ for $\bar{x} \in \overline{O}$, it is clear that the minimal projections $e_{x_0}, x_0 \in X_0$, of $K$ correspond in a 1-1 onto manner to the collection $e_{x_0}, \bar{x}_0 \in \overline{X}_0$, of all characteristic functions on single point, open-closed sets of $\overline{X}$, that $X_0$ and $\overline{X}_0$ may be identified, that (by condition 1) the points of $\overline{X}_0$ are dense in $\overline{X}$, and that (by condition (2)) each nonempty subset of points in $\overline{X}_0$ is contained in an open-closed subset of $\overline{X}$ which contains all and only those points of $\overline{X}_0$ that are in the given subset. It follows then by Theorem 1 that $C(\overline{X})$ and hence $K$ is isomorphic to a $B(X_0^*)$ wherein the set $X_0^*$ may be identified with $\overline{X}_0$ and thus with $X_0$.  

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A NOTE ON UNSTABLE HOMEOMORPHISMS¹

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In [1] W. R. Utz introduced the concept of an unstable² homeomorphism and raised the question of whether there exists an unstable homeomorphism of a compact continuum onto itself. In this note an example of such an homeomorphism will be given.

Let $C$ denote the complex unit circle and for each $z \in C$, let $g(z) = z^2$. Then $g : C$ onto $C$ determines an inverse limit space $Σ_2 = \{(a_0, a_1, a_2, \cdots)\}$ for each non-negative integer $i$, $a_i \in C$ and $g(a_{i+1}) = a_i$. For $a, b \in Σ_2$, the function $ρ(a, b) = \sum_{i=0}^{\infty} |a_i - b_i| / 2^i$ is a metric for $Σ_2$; $Σ_2$ is familiar as the “two-solenoid,” and is a compact, indecomposable continuum. Define $f : Σ_2$ onto $Σ_2$ as follows: for each $a = (a_0, a_1, \cdots) \in Σ_2$, let $f(a) = [g(a_0), g(a_1), \cdots]$. Then $f(a) = (a_0^2, a_1^2, \cdots) = (a_0^2, a_0, a_1, \cdots)$, $f^{-1}(a) = (a_1, a_2, a_3, \cdots)$, and $f$ is a homeomorphism of $Σ_2$ onto $Σ_2$.

To show that $f$ is unstable, suppose that $a = (a_0, a_1, \cdots)$ and $b = (b_0, b_1, \cdots)$ are distinct points of $Σ_2$. Consider, as Case 1, that $a_0 \neq b_0$. Let $e^{iθ} = a_0$, $e^{iφ} = b_0$, where $0 ≤ θ, φ < 2π$. Then there exists a non-negative integer $n$ such that the angle between the terminal rays of $2^nθ$ and $2^nφ$ is greater than $π/2$. Then $ρ[f^n(a), f^n(b)] ≥ |a_0^{2^n} - b_0^{2^n}| = |e^{i2^nθ} - e^{i2^nφ}| > 1$.

Case 2: for some integer $n > 0$, $a_n \neq b_n$, but $a_i = b_i$, for $0 ≤ i < n$. Then $f^{-n}(a) = (a_n, a_{n+1}, a_{n+2}, \cdots)$, $f^{-n}(b) = (b_n, b_{n+1}, b_{n+2}, \cdots)$, and there-

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² A homeomorphism $f$ of a compact metric space $X$ onto $X$ is said to be unstable provided there exists a fixed positive number $δ$, such that if $x$ and $y$ are distinct points of $X$, then there exists an integer $n$, such that $ρ[f^n(x), f^n(y)]$ is greater than $δ$. Presented to the Society, November 26, 1954; received by the editors June 25, 1954.