1. Let \( p \) be a prime \( \geq 3 \). If \((r, p) = 1\), define \( r' \) by means of \( rr' \equiv 1 \pmod{p} \); the symbol \( R(r) \) will denote the least positive residue of \( r \pmod{p} \). Following Maillet we define the determinant \( D_p \) by means of
\[
D_p = \left| R(rs') \right| \quad (r, s = 1, \ldots, (p - 1)/2).
\]
Maillet raised the question whether \( D_p \neq 0 \) for all \( p \). Malo computed \( D_p \) for several small values of \( p \):
\[
D_3 = 1, \quad D_5 = -5, \quad D_7 = 7^2, \quad D_{11} = 11^4, \quad D_{13} = -13^5,
\]
and conjectured that generally
\[
D_p = (-p)^{(p-3)/2}.
\]
For references see [2, pp. 340–342].

Making use of the easily proved transformation
\[
D_p = (-p)^{(p-3)/2} \left( \frac{r's'}{p} \right) \quad (r, s = 2, \ldots, (p - 1)/2),
\]
which is obtained by subtracting \( r \) times the first row of \( D_p \) from the \( r \)th row, it is evident that \( D_p \) is indeed divisible by the power of \( p \) indicated in (1.2). In turn (1.3) may be further simplified by successive row subtractions to
\[
D_p = (-p)^{(p-3)/2} \left( \frac{r's'}{p} - \frac{(r-1)s'}{p} \right) \quad (r, s = 2, \ldots, (p - 1)/2).
\]
It is also not difficult to show that
\[
D_p = \pm \left| p + R(rs) \right| \quad (r, s = 1, \ldots, (p - 1)/2)
\]
which in turn reduces to
\[
D_p = \pm p^2 \left( \frac{r's'}{p} - \frac{(r-1)s'}{p} \right) \quad (r, s = 3, \ldots, (p - 1)/2).
\]
This formula is particularly convenient for computation. Note that

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the elements in the determinants in (1.4) and (1.6) consist only of zeros and ones.

By means of (1.6) it is not difficult to verify that (1.2) holds for \( p = 17 \) and 19 but not for \( p = 23 \); in the last case an additional factor 3 occurs. Thus Malo's conjecture is not correct. We shall however show that \( D_p \) never vanishes. This is a consequence of the formula proved below:

\[
D_p = \pm p^{(p-3)/2} \cdot h,
\]

where \( h \) denotes the first factor of the class number of the cyclotomic field \( k(\zeta p) \).

Some related determinants are discussed briefly in §3.

2. Put

\[
D_p(x) = \left| x + R(rs') \right| \quad (r, s = 1, \ldots, \frac{p-1}{2}),
\]

so that \( D_p(0) = D_p \). Since the last column of \( D_p \) consists of the numbers \( p-2, p-4, \ldots, 1 \), it follows that by addition of twice the first column to the last column of both \( D_p \) and \( D_p(x) \) we get

\[
D_p(x) = \frac{3x + p}{p} D_p.
\]

If we take \( x = -p/2 \) then (2.2) becomes

\[
D'_p = -D_p/2,
\]

where

\[
D'_p = D_p(-p/2) = \left| \{rs'\} \right| \quad (r, s = 1, \ldots, \frac{p-1}{2})
\]

and

\[
\{r\} = R(r) - p/2.
\]

Note that (2.5) implies

\[
\{-r\} = -\{r\}.
\]

In the next place let \( g \) denote a primitive root (mod \( p \)) and put

\[
D''_p = \left| \{g^{i-j}\} \right| \quad (i, j = 0, \ldots, \frac{p-3}{2}).
\]

Except for sign and order, the numbers

\[
\{1\}, \{g\}, \ldots, \{g^{(p-3)/2}\}
\]

are the same as the numbers \( \{1\}, \{2\}, \ldots, \{(p-1)/2\} \). Conse-
Subsequently comparison of (2.4) and (2.7) shows that

(2.8) \[ D_p' = \pm D_p'' \]

Since \( g^{(p-1)/2} = -1 \), it follows from (2.6) that \( D_p'' \) is not a circulant. However, if \( \alpha \) denotes a primitive \((p-1)\)th root of unity, then clearly

(2.9) \[ D_p''' = \left| \{ \alpha^{i-j} \} \right| \quad (i, j = 0, \ldots, (p - 1)/2) \]

is a circulant; moreover it is evident that

(2.10) \[ D_p'' = D_p''' \]

Using the familiar formula for a circulant, (2.9) yields

(2.11) \[ D_p''' = \prod_{i=0}^{(p-3)/2} \sum_{j=0}^{(p-3)/2} \{ g^{i-j} \} \alpha^{i(2j+1)} \]

Now on the other hand the first factor of the class number of \( k(e^{2\pi i/p}) \) is given by [3, p. 35]

(2.12) \[ h = (2p)^{-\frac{(p-3)}{2}} \prod_{i=0}^{(p-3)/2} \phi(\alpha^{2i+1}), \]

where

\[ \phi(x) = \sum_{i=0}^{p-2} R(g^i) x^i. \]

Then if \( \beta = \alpha^{2i+1} \),

\[ \phi(\beta) = \sum_{i=0}^{(p-3)/2} R(g^i) \beta^i - \sum_{i=0}^{(p-3)/2} R(-g^i) \beta^i \]

\[ = \sum_{i=0}^{(p-3)/2} (2R(g^i) - \rho) \beta^i = 2 \sum_{i=0}^{(p-3)/2} \{ g^i \} \beta^i. \]

Hence using (2.3), (2.8), (2.10), (2.11), and (2.12) we get

(2.13) \[ h = \pm p^{-\frac{(p-3)}{2}} D_p, \]

which proves (1.7).

It follows at once from (2.13) that

(2.14) \[ D_p \neq 0 \]

for all \( p \geq 3 \). On the other hand Kummer’s criterion [3, p. 35] for divisibility of \( h \) by \( p \) shows that
(2.15) \[ p^{(p-1)/2} \mid D_p \]

if and only if \( p \) divides the numerator of one of the Bernoulli numbers \( B_2, B_4, \cdots, B_{p-3} \); moreover we can assert that (2.15) holds for infinitely many primes \( p \).

3. By a formula of Eisenstein

\[
\left[ \frac{m}{n} \right] = \frac{m}{n} - \frac{1}{2} + \frac{1}{2n} \sum_{k=1}^{n-1} \sin \frac{2km\pi}{n} \cot \frac{k\pi}{n}.
\]

This implies

\[
\{m\} = -\sum_{k=1}^{(p-1)/2} \sin \frac{2km\pi}{p} \cot \frac{k\pi}{p},
\]

where \( \{m\} \) is defined by (2.5). Replacing \( m \) by \( rs' \) we get

\[
\{rs'\} = -\sum_{k=1}^{(p-1)/2} \sin \frac{2krs\pi}{p} \cot \frac{ks\pi}{p},
\]

and therefore

\[
D'_p = (-1)^{(p-1)/2} \left| \sin \frac{2rs\pi}{p} \right| \left| \cot \frac{rs\pi}{p} \right|,
\]

where in each determinant \( r, s = 1, \cdots, (p-1)/2 \). Now since

\[
\left| \sin 2rs\pi/p \right|^2 = 2^{-(p-1)}p^{(p-1)/2},
\]

it follows, using (2.3), that \( \left| \cot rs\pi/p \right| = \pm 2^{(p-3)/2}p^{-(p-1)/4}D_p \). Hence (2.13) yields

\[
(3.1) \quad \left| \cot rs\pi/p \right| = \pm 2^{(p-3)/2}p^{-(p-5)/4}h.
\]

Again, recalling the definition of the Dedekind sum

\[
s(r, p) = \sum_{k=1}^{p-1} \frac{k}{p} \left( \frac{rk}{p} - \left[ \frac{rk}{p} \right] - \frac{1}{2} \right) = \frac{1}{p^2} \sum_{k=1}^{p-1} k\{rk\}
\]

and Rademacher's formula (for an elementary proof see [1])

\[
s(r, p) = \frac{1}{4p} \sum_{k=1}^{p-1} \cot \frac{rk\pi}{p} \cot \frac{k\pi}{p},
\]

we see that

\[
(3.2) \quad s(rs', p) = \frac{1}{2p} \sum_{k=1}^{(p-1)/2} \cot \frac{rk\pi}{p} \cot \frac{ik\pi}{p},
\]
where as above \( tt' \equiv 1 \pmod{p} \). If we put

\[
\Delta_p = \left| s(r t', p) \right| \quad (r, t = 1, \cdots, (p - 1)/2),
\]

it follows from (3.2) that

\[
\Delta_p = (2p)^{-\frac{(p-1)}{2}} \left| \cot \frac{r t \pi}{p} \right|^2.
\]

Consequently by (3.1)

(3.3) \[ \Delta_p = 2^{(p-6)/2} p^{-2} h^2. \]

*Added in proof.* Professor S. Chowla has kindly informed the writers that he and A. Weil had proved the formula (1.7) several years ago but had not published the result.

**References**


**Duke University**