1. Introduction. Let $X$ be any set, let $S$ be a Boolean $\sigma$-algebra of subsets of $X$, and let $F$ be a function from $S$ to non-negative operators on a Hilbert space $\mathcal{H}$ such that $F(X) = 1$ and $F$ is countably-additive in the weak operator topology. Neumark \cite{2} has shown that there exists a Hilbert space $\mathcal{K}$ of which $\mathcal{H}$ is a subspace and a spectral measure $E$ defined on $S$ such that $F(S)P = PE(S)P$ for all $S$ in $S$, where $P$ is projection of $\mathcal{K}$ on $\mathcal{H}$. Let us rephrase this situation so that we speak of algebras rather than Boolean algebras and linear functions rather than measures. Thus, we consider, instead of the Boolean $\sigma$-algebra $S$, the $C^*$-algebra $\mathcal{A}$ of all bounded functions on $X$ which are measurable with respect to $S$. A $C^*$-algebra is defined as a complex Banach algebra with an involution $x \mapsto x^*$ such that $\|xx^*\| = \|x\|^2$ for all $x$ in the algebra. The measure $F$ is supplanted by the linear function $\mu$ on $\mathcal{A}$

$$\mu(f) = \int f(\gamma)dF(\gamma), \quad f \in \mathcal{A},$$

where the integral is to be taken in the weak sense. The theorem now asserts that $\mu(f)P = P\rho(f)P$, where

$$\rho(f) = \int f(\gamma)dE(\gamma), \quad f \in \mathcal{A}.$$

In the original formulation, $E$ was an improvement over $F$ because $E$ was a spectral measure; in the reformulation, $\rho$ is an improvement over $\mu$ since $\rho$ is a *-homomorphism. When the situation is phrased in this manner, the question naturally occurs: "Is it essential that the algebra $\mathcal{A}$ be commutative?" The present paper is devoted to a discussion of this point.

2. The main theorem. If $\mathcal{A}$ and $\mathcal{B}$ are $C^*$-algebras and $\mu$ is a linear function from $\mathcal{A}$ to $\mathcal{B}$, we shall say that $\mu$ is positive if $\mu(A) \geq 0$ whenever $A \in \mathcal{A}$ and $A \geq 0$. The algebra of $n \times n$ matrices with entries in $\mathcal{A}$ is also a $C^*$-algebra, which we shall denote by $\mathcal{A}^{(n)}$. By applying $\mu$ to each entry of an element of $\mathcal{A}^{(n)}$, we obtain an element of $\mathcal{B}^{(n)}$; this linear function from $\mathcal{A}^{(n)}$ to $\mathcal{B}^{(n)}$ will be denoted by $\mu^{(n)}$. We shall say that $\mu$ is completely positive if $\mu^{(n)}$ is positive for each positive integer $n$.  

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Theorem 1. Let \( \mathcal{A} \) be a C*-algebra with a unit, let \( \mathcal{K} \) be a Hilbert space, and let \( \mu \) be a linear function from \( \mathcal{A} \) to operators on \( \mathcal{K} \). Then a necessary and sufficient condition that \( \mu \) have the form

\[
\mu(A) = V^* \rho(A) V
\]

for all \( A \in \mathcal{A} \), where \( V \) is a bounded linear transformation from \( \mathcal{K} \) to a Hilbert space \( K \) and \( \rho \) is a *-representation of \( \mathcal{A} \) into operators on \( K \), is that \( \mu \) be completely positive.

Proof of necessity. Suppose that \( \mu(A) = V^* \rho(A) V \). Let \( M = (A_{ij}) \) be a non-negative matrix in \( \mathcal{A}^{(n)} \). Then \( \mu(A_{ij}) \) is an operator on the direct sum of \( \mathcal{K} \) with itself \( n \) times. What we have to check is that

\[
\sum_{i,j} (\mu(A_{ij}) x_j, x_i) \geq 0
\]

whenever \( x_1, \ldots, x_n \) are vectors in \( \mathcal{K} \). Since \( \rho \) is a *-representation, the matrix \( (\rho(A_{ij})) \) is a non-negative operator on \( K \oplus \cdots \oplus K \); and therefore,

\[
\sum_{i,j} (\mu(A_{ij}) x_j, x_i) = \sum_{i,j} (\rho(A_{ij}) V x_j, V x_i) \geq 0.
\]

Proof of sufficiency. Suppose that \( \mu \) is completely positive. Consider the vector space \( \mathcal{A} \otimes \mathcal{K} \), the algebraic tensor product of \( \mathcal{A} \) and \( \mathcal{K} \). For \( \xi = \sum_i A_i \otimes x_i \) and \( \eta = \sum_j B_j \otimes y_j \) in \( \mathcal{A} \otimes \mathcal{K} \) we define

\[
(\xi, \eta) = \sum_{i,j} (\mu(B_j A_i) x_i, y_j).
\]

Since \( \mu \) was assumed to be completely positive, it follows that

\[
(\xi, \xi) = \sum_{i,j} (\mu(A_j^* A_i) x_i, x_j) \geq 0.
\]

Hence \( \langle \cdot, \cdot \rangle \) is a positive Hermitian bilinear form. There is a natural mapping \( \rho' \) from \( \mathcal{A} \) to linear transformations on \( \mathcal{A} \otimes \mathcal{K} \) given by

\[
\rho'(A) \sum_i B_i \otimes y_i = \sum_i (AB_i) \otimes y_i.
\]

We shall show that for all \( A \) in \( \mathcal{A} \) and \( \xi \) in \( \mathcal{A} \otimes \mathcal{K} \)

\[
(\rho'(A) \xi, \rho'(A) \xi) \leq \|A\|^2 (\xi, \xi).
\]

If (1) were not universally true, we could find \( A \) in \( \mathcal{A} \) and

\[
\xi = \sum_i B_i \otimes x_i
\]

in \( \mathcal{A} \otimes \mathcal{K} \) such that \( (\xi, \xi) \leq 1 \) and \( \|A\| < 1 \), but \( (\rho'(A) \xi, \rho'(A) \xi) > 1 \).
Then \((\rho'(A'*A)\xi, \xi) > 1\), which implies, by the Schwarz inequality, that \((\rho'(A'*A)\xi, \rho'(A'*A)\xi) > 1\). By continuing in this manner, we find that

\[
(\rho'([A*A]^k)\xi, \xi) > 1 \quad \text{for } k = 1, 2, \cdots.
\]

Since \(\mu\) is positive and 
\[-\|C\|_1 \leq C \leq \|C\|_1\]
whenever \(C\) is a self-adjoint element of \(\mathcal{A}\), it follows that 
\[-\|C\|\mu(1) \leq \mu(C) \leq \|C\|\mu(1) \]
and hence that 
\[\|\mu(C)\| \leq \|C\|\|\mu(1)\| ;
\] this inequality shows that \(\mu\) is uniformly continuous on the self-adjoint elements of \(\mathcal{A}\), and it is easy to see from this that \(\mu\) must be uniformly continuous on all of \(\mathcal{A}\). The uniform continuity of \(\mu\) together with the fact that \(\|A\| < 1\) shows that

\[
(\rho'([A*A]')\xi, \xi) = \sum_{i,j} (\mu(B^*_j[A*A]^{*k}B_i)x_i, x_j)
\]

converges to 0. This contradiction proves (1).

Let \(\mathcal{N}\) be the set of all \(\xi\) in \(\mathcal{A} \otimes \mathcal{K}\) such that \((\xi, \xi) = 0\). By the Schwarz inequality, \(\mathcal{N}\) is a linear manifold and by (1), \(\mathcal{N}\) is invariant under \(\rho'(\mathcal{A})\). Therefore, the quotient space \(\mathcal{A} \otimes \mathcal{K}/\mathcal{N}\) is a pre-Hilbert space, and each \(A\) in \(\mathcal{A}\) naturally induces a bounded operator on the completion \(\mathcal{K}\) of \(\mathcal{A} \otimes \mathcal{K}/\mathcal{N}\). Let

\[Vx = 1 \otimes x + \mathcal{N}\]

for all \(x \in \mathcal{K}\). Then \(\|Vx\|^2 \leq (\mu(1)x, x)\), so that \(V\) is a bounded linear transformation from \(\mathcal{K}\) to \(\mathcal{K}\). Now

\[(V^*\rho(A)Vx, y) = (\rho(A)Vx, Vy) = (\rho'(A)1 \otimes x, 1 \otimes y) = (A \otimes x, 1 \otimes y) = (\mu(A)x, y)\]

for all \(x\) and \(y\) in \(\mathcal{K}\), and therefore \(\mu(A) = V^*\rho(A)V\).

3. Remarks. The operator \(\rho(1)\) is a projection. We can take \(\rho(1)\) to be 1, for we can replace \(V\) by \(\rho(1)V\) and then replace the space \(\mathcal{K}\) by the subspace \(\rho(1)\mathcal{K}\). Assuming that this has been done, we have \(\mu(1) = V^*V\), so that if \(\mu(1) = 1\), then \(V\) is an isometry. Since an isometry can be considered as an embedding, the Neumark theorem follows from Theorem 1, provided we can show that when \(\mathcal{A}\) is commutative, positivity of \(\mu\) implies complete positivity. This fact will be proved in the next section.

It might be thought possibly that positivity always implies complete positivity. We give a counter-example to show that this is not the case. Let \(\mathcal{A}\) be the algebra of \(n \times n\) matrices with complex entries. We denote by \(e_{ij}\) the matrix with 1 in the \(i\)th row and \(j\)th column and
with zeros elsewhere. The $e_{ij}$'s are a basis for the vector space $\mathcal{A}$. We define a linear function $\mu$ from $\mathcal{A}$ to $\mathcal{A}$ by specifying the values of $\mu$ on this basis: $\mu(e_{ij})$ is to be a matrix whose $(r, s)$th entry is $(r - j)(s - i)$. Now $\mu$ is positive; for let $A = (a_{ij}) \geq 0$ be an $n \times n$ matrix and let

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

be any vector. Then

$$(\mu(A)x, x) = \sum_{i,j,r,s} a_{ij}(r - j)(s - i)x_rx_s$$

$$= \sum_{i,j} a_{ij} \left( \sum_r (r - j)x_r \right) \left( \sum_s (s - i)x_s \right) \geq 0$$

since $A \geq 0$.

On the other hand, $\mu$ is not completely positive, for we shall show that $\mu^{(n)}$ is not positive, $n$ being as above. Let $E$ be the element of $\mathcal{A}^{(n)}$ whose $(i, j)$th entry is the matrix $e_{ij}$. It is easily seen that $E \geq 0$. But $\mu^{(n)}(E)$ is not $\geq 0$, for

$$\sum_{i,j,r,s} (r - j)(s - i)\delta_{ij}\delta_{rs} = \sum_{i,j} (i - j)(j - i) < 0.$$


Theorem 2. If $\mathcal{A}$ is a $W^*$-algebra of finite type (see [1]), then the center-valued trace $t$ is completely positive.

Proof. Suppose $M = (A_{ij})$ is a non-negative matrix in $\mathcal{A}^{(n)}$ and suppose $\epsilon > 0$. It is shown in [1] that there exist a finite number $U_1, \ldots, U_r$ of unitary operators in $\mathcal{A}$ and non-negative numbers $\alpha_1, \ldots, \alpha_r$ with $\sum_k \alpha_k = 1$ such that

$$\| t(A_{ij}) - \sum_k \alpha_k U_k^*A_{ij}U_k \| < \epsilon \quad \text{for } i, j = 1, \ldots, n.$$

But then if $x_1, \ldots, x_n$ are any vectors,

$$\left| \sum_{i,j} (t(A_{ij})x_j, x_i) - \sum_k \alpha_k (U_k^*A_{ij}U_kx_j, x_i) \right| \leq \epsilon \sum_{i,j} |(x_j, x_i)|$$

and it is clear that

$$\sum_{i,j,k} \alpha_k (U_k^*A_{ij}U_kx_j, x_i) = \sum_k \alpha_k \sum_{i,j} (A_{ij}U_kx_j, U_kx_i) \geq 0$$

since $M \geq 0$. It follows that $t^{(n)}(M) \geq 0$. 

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Theorem 3. If $\mathcal{A}$ is a $C^*$-algebra and $\mu$ is a positive linear function with complex values, then $\mu$ is completely positive.

Proof. Suppose $M = (A_{ij})$ is a non-negative matrix in $\mathcal{A}^{(n)}$, and $\{\lambda_i\}$ are complex numbers. We wish to check that

$$\sum_{i,j} \mu(A_{ij}) \lambda_j \lambda_i \geq 0.$$  

But

$$\sum_{i,j} \mu(A_{ij}) \lambda_j \lambda_i = \mu(\sum_{i,j} A_{ij} \lambda_j \lambda_i).$$

Writing $M = N^*N$ with $N = (B_{ij})$, we see that

$$\sum_{i,j} A_{ij} \lambda_j \lambda_i = \sum_{i,j} \lambda_j \lambda_i \sum_k B_{ki}^* B_{kj}$$

$$= \sum_k \left( \sum_i \lambda_i B_{ki} \right)^* \left( \sum_i \lambda_i B_{ki} \right) \geq 0$$

and so $\mu(\sum_{i,j} A_{ij} \lambda_j \lambda_i) \geq 0$ since $\mu$ is positive.

Theorem 3 together with Theorem 1 gives the known fact that a state of a $C^*$-algebra induces a representation [3].

Theorem 4. If $\mathcal{A}$ is a commutative $C^*$-algebra and $\mu$ is a positive operator-valued linear function on $\mathcal{A}$, then $\mu$ is completely positive.

Proof. We may take $\mathcal{A}$ as the algebra of all continuous complex-valued functions vanishing at $\infty$ on a locally compact Hausdorff space $\Gamma$. Let $M = (f_{ij})$ be a non-negative matrix in $\mathcal{A}^{(n)}$. If $x_1, \cdots, x_n$ are vectors in the Hilbert space, we wish to verify that

$$\sum_{i,j} (\mu(f_{ij}) x_j, x_i) \geq 0.$$  

By the Riesz-Markoff theorem, there exists a regular measure $m$ on $\Gamma$ such that $\sum_i (\mu(f) x_i, x_i) = \int_{\Gamma} dm$ for all $f \in \mathcal{A}$. Then by the Riesz-Markoff and Radon-Nikodym theorems, there exist measurable functions $h_{ij}$ such that

$$(\mu(f) x_j, x_i) = \int_{\Gamma} f h_{ij} dm \quad \text{for all } f \in \mathcal{A}.$$  

Now the matrix $(h_{ij}(\gamma))$ is non-negative almost everywhere; for

$$\int_{\Gamma} f \sum_{i,j} h_{ij} \lambda_i \lambda_j dm \geq 0 \quad \text{for all } f \geq 0 \in \mathcal{A},$$  

and hence $\sum_{i,j} h_{ij}(\gamma) \lambda_i \lambda_j \geq 0$ for all $r$-tuples $\lambda_1, \cdots, \lambda_r$ of complex numbers with rational real and imaginary parts and for all $\gamma \in \Gamma$ with the exception of $\gamma \in N$ where $m(N) = 0$. Also the matrix $(f_{ij}(\gamma))$ is
non-negative for all $\gamma \in \Gamma$ by Theorem 3. Therefore,
\[ \sum_{i,i} f_{i,i}(\gamma) h_{i,i}(\gamma) \geq 0 \quad \text{almost everywhere} \]
and so
\[ \sum_{i,i} (\mu(f_{i,j}) x_j, x_i) = \int_{\Gamma} \sum_{i,i} f_{i,i} h_{i,i} dm \geq 0. \]

**BIBLIOGRAPHY**


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