In this paper I discuss integrals which provide explicit asymptotic expansions of generalized basic hypergeometric functions. The problem of asymptotic expansions for hypergeometric functions has been considered previously by C. S. Majer [2], E. M. Wright [5], and, for basic functions, by G. N. Watson [4].

Let
\[
\sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \cdots (a_A)_n z^n}{(b_1)_n(b_2)_n \cdots (b_B)_n n!}
\]

where \((a)_n = a(a+1)(a+2) \cdots (a+n-1)\). Also let
\[
\Gamma[(a); (b)] = \frac{\Gamma(a_1) \Gamma(a_2) \cdots \Gamma(a_A)}{\Gamma(b_1) \Gamma(b_2) \cdots \Gamma(b_B)}
\]

A dash will denote the omission of a vanishing factor in a sequence. Thus, \((a)'-a_r\) denotes the sequence \(a_1-a_r, a_2-a_r, \ldots, a_{r-1}-a_r, a_{r+1}-a_r, \ldots, a_A-a_r\).

It is known already (see [2] and [5]) that if
\[
I(z) = \frac{1}{2\pi i} \int_{-\infty}^{i\infty} \Gamma[(a)+s, (b)-s] z^s ds,
\]

\[
\sum A(z) = \sum_{\mu=1}^{A} z^{-a_\mu} \Gamma[(a)'-a_\mu, (b)+a_\mu] \times_{B+cF_{A+D-1}} \left[ \begin{array}{c} b + a_\mu, 1 + a_\mu - (c); \\ 1 + a_\mu - (a), (d) + a_\mu; \end{array} \right] (-1)^{A+c z^{-1}}
\]

\[
\sum B(z) = \sum_{\nu=1}^{B} z^{b_\nu} \Gamma[(b)+b_\nu, (b)'+-b_\nu] \times_{A+dF_{B+C-1}} \left[ \begin{array}{c} a + b_\nu, 1 + b_\nu - (d); \\ 1 + b_\nu - (b), (c) + b_\nu; \end{array} \right] (-1)^{B+d z^{-1}}
\]

then, provided that \(|A + B - C - D|/2 > |\arg z|\),

\[(1.1) \quad I(z) = \sum A(z) \sim \sum B(z) \quad \text{when } B + C < A + D,
\]

\[(1.2) \quad I(z) = \sum B(z) \sim \sum A(z) \quad \text{when } B + C > A + D,
\]
and (iii) \( I(1) = \sum_A(1) = \sum_B(1) \), when \( A - C = B - D \geq 0 \) and

\[
R1 \sum (c + d - a - b) > 0.
\]

In particular, the cases \( A = 1, B = 2, C = D = 0 \), and \( A = B = C = 1, D = 0 \), lead to

\[
\begin{align*}
\text{I} \text{F} \text{I} [a; b; z] & \sim \Gamma [1 + a - b; 1 - b] z^{-a} \text{F} \text{O} [a, 1 + a - b; 1/z] + e^{z} z^{a-b} \Gamma [b; a] \text{F} \text{O} [1 - a, b - a; 1/z]
\end{align*}
\]

provided that \( |\text{arg } z| < \pi/2 \). This is Barnes’ well-known result for the confluent hypergeometric function (see [1]).

2. The analogue for basic functions. I shall now state the corresponding results for basic hypergeometric functions together with an outline of the proof. In the usual notation for basic series, let

\[
\left. \text{I} \text{F} \text{I}_\text{B} [(a); (b); z] = \sum_{n=0}^{\infty} \frac{(a)_n (a_2)_{n} \cdots (a_A)_{n a_n}}{(b)_n (b_2)_{n} \cdots (b_B)_{n}(1)_{n}} \right.
\]

where \( (a)_n = (1 - q^a)(1 - q^{a+1}) \cdots (1 - q^{a+n-1}) \), and \( |q| < 1 \). Also let

\[
\prod_{n=0}^{P} [(a); (b)] = \prod_{n=0}^{P} \frac{(1 - q^{a+n})(1 - q^{a+n+1}) \cdots (1 - q^{a+n})}{(1 - q^{b+n})(1 - q^{b+n+1}) \cdots (1 - q^{b+n})}.
\]

Let \( I_{P, R} \) be the integral

\[
\frac{1}{2\pi i} \int \prod_{n=0}^{P} [(a) + s, (b) - s, 1 - z + s, z - s; (c) + s, (d) - s] ds
\]

taken round the contour \( A(-i\pi/t)B(i\pi/t)C(R+i\pi/t)D(R-i\pi/t) \), and let \( I_{P, R'} \) be the same integral taken round the contour \( A(-i\pi/t)B(i\pi/t)E(-R'+i\pi/t)F(-R'-i\pi/t) \). We shall assume now that \( P > R \) and \( P > R' \), and that both contours are indented so that (assuming that \( R \) and \( R' \) are integers) the first \( R \) of every ascending sequence of poles of \( \prod_{n=0}^{P} (s) \) fall inside \( ABCD \), and the first \( R' \) of every descending sequence of poles of \( \prod_{n=0}^{P} (s) \) fall inside \( ABEF \). We assume also that \( q = r^{-t}, t > 0 \), though the restriction that \( q \) is real can easily be removed from the final result by analytic continuation over the circle \( |q| < 1 \).

By the periodicity of the integrand, we have

\[
\int_{BC} \prod_{n=0}^{P} (s) ds = \int_{AD} \prod_{n=0}^{P} (s) ds \quad \text{and} \quad \int_{FA} \prod_{n=0}^{P} (s) ds = \int_{EB} \prod_{n=0}^{P} (s) ds.
\]
Hence

$$I_{P,R} = \int_{AB} \prod_{s=1}^{P} I(s) ds + \int_{CD} \prod_{s=1}^{P} I(s) ds,$$

and

$$-I_{P,R'} = \int_{AB} \prod_{s=1}^{P} I(s) ds + \int_{BF} \prod_{s=1}^{P} I(s) ds.$$

But

$$I_{P,R} = \sum \text{(residues within } ABCD \text{ of } \prod_{s=1}^{P} I(s)),
= \frac{1}{l} \sum_{\mu=1}^{D} \prod_{s=1}^{P} \left[ (a + d_{\mu}, (b) - d_{\mu}, 1 - z + d_{\mu}, z - d_{\mu}) \right]
\times \sum_{n=0}^{R} \frac{((c) + d_{\mu})_{n}(1 + d_{\mu} - (b))_{n}Q^{n}}{(c)_{n}(1 + d_{\mu} - (b))_{n}}$$

where $Q = (-q^{(n+1)/2+d_{\mu}})_{(D-B)q^{b_{1}+\cdots+b_{B}+d_{1}+\cdots+d_{B}+z}},$ that is

$$I_{P,R} = \sum_{\mu=1}^{D} \sum_{n=0}^{R} \text{(d)}, \text{ say.}$$

Similarly,

$$-I_{P,R'} = \frac{1}{l} \sum_{\nu=1}^{C} \prod_{s=1}^{P} \left[ (a + c_{\nu}, (a) - c_{\nu}, z + c_{\nu}, 1 - z - c_{\nu}) \right]
\times \sum_{n=0}^{R'} \frac{((d) + c_{\nu})_{n}(1 + c_{\nu} - (a))_{n}Q'^{n}}{(d)_{n}(1 + c_{\nu} - (a))_{n}}$$

where $Q' = (-q^{(n+1)/2+c_{\nu}})_{(C-A)q^{a_{1}+\cdots+a_{A}-c_{1}+\cdots-c_{0}+1-z}},$ that is,

$$-I_{P,R'} = \sum_{\nu=1}^{C} \sum_{n=0}^{R'} \text{(c)}, \text{ say,}$$

and so

$$\int_{AB} \prod_{s=1}^{P} I(s) ds = \sum_{\mu=1}^{D} \prod_{s=1}^{P} \sum_{n=0}^{R} \text{(d)} + \int_{DC} \prod_{s=1}^{P} I(s) ds
= \sum_{\nu=1}^{C} \prod_{s=1}^{P} \sum_{n=0}^{R'} \text{(c)} + \int_{BF} \prod_{s=1}^{P} I(s) ds.$$

(2.1)

Now

$$\left| \int_{DC} \prod_{s=1}^{P} I(s) ds \right| \leq \frac{1}{2\pi} \int_{-\gamma/\ell}^{\gamma/\ell} \left| \prod_{s=1}^{P} (R + i\tau) \right| d\tau,$$
and
\[ \left| \int_{\mathcal{P}} \prod_{s} (s) ds \right| \leq \frac{1}{2\pi} \int_{-\epsilon/1}^{\epsilon/1} \prod_{s} (-R' + iv) \, dv. \]

It has been shown previously (Slater [3]) that if \( D=B \) and \( \mathcal{R} (b-d) > 0 \), or if \( D>B \), \( \int_{\mathcal{A}} \prod_{s} (s) ds = \sum_{D} \prod_{s} \sum_{R} (d) \). Also if \( A=C \) and \( \mathcal{R} (a-c) > 0 \), or if \( C>A \),
\[ \int_{\mathcal{A}} \prod_{s} (s) ds = \sum_{C} \prod_{s} \sum_{(c)} (c). \]

In all cases, even when \( C<A \), or when \( D<B \), we have for \( R \) fixed
\[ \left| \int_{\mathcal{D}} \prod_{s} (s) ds \right| \leq \frac{1}{t} \prod_{1}^{\infty} \frac{1+q^{(a)+R+n}}{1-q^{(c)+R+n}} \frac{1+q^{(b)-R+n}}{1-q^{(d)-R+n}}. \]

But the next term of the series \( \sum_{D} \prod_{s} \sum_{R} (d) \) would be
\[ \sum_{\mu=1}^{D} \prod_{(a)} + d_{\mu}, 1 - z + d_{\mu}, (b) - d_{\mu}, z - d_{\mu} \]
\[ (c) + d_{\mu}, (d)' - d_{\mu}, 1 \]
\[ \times \frac{(c) + d_{\mu} R+1 (1 + d_{\mu} - (b)) R+1 Q_{R+1}^{(a)}}{(a) + d_{\mu} R+1 (1 + d_{\mu} - (d)) R+1} \]
which is of the same order in \( R \) as \( \int_{\mathcal{D}} \prod_{s} (s) ds \). Similarly, for \( R' \) fixed \( \int_{\mathcal{P}} \prod_{s} (s) ds \) is also bounded above as \( P \to \infty \), and this integral is of the same order in \( R' \) as the \( (R'+1) \)th term of the series \( \sum_{c} \prod_{s} \sum_{R'} (c) \).

Hence we have
\[ \frac{1}{2\pi i} \int_{-\epsilon/1}^{\epsilon/1} \prod_{1}^{\infty} [(a) + s, 1 - z + s, (b) - s, z - s; (c) + s, (d) - s] ds \]
\[ \sim \frac{1}{t} \sum_{\mu=1}^{D} \prod_{(a)} + d_{\mu}, 1 - z + d_{\mu}, (b) - d_{\mu}, z - d_{\mu} \]
\[ (c) + d_{\mu}, (d)' - d_{\mu}, 1 \]
\[ \times B_{c+c} \Phi_{A+D-1} \left[(a) + d_{\mu}, 1 + d_{\mu} - (b) + (c) + d_{\mu}, (d)'; Q \right] \]
\[ \times A_{c+c} \Phi_{B+c-1} \left[(d) + c_{r}, 1 + c_{r} - (a); Q' \right] \]
\[ (2.2) \]
where

\[ Q = (-q^{(n+1)/2} + d_\alpha) (D - B) q^{b_1 + \cdots + b_B - d_1 - \cdots - d_D + z}, \]

and

\[ Q' = (-q^{(n+1)/2 + c_\gamma}) (C - A) q^{a_1 + \cdots + a_A - c_1 - \cdots - c_C + 1 - z}, \]

even when \( D < B \) or when \( C < A \).

In particular, let

\[ A = B = 0, \quad C = 1, \quad D = 2, \]

then

\[
\frac{1}{2\pi i} \int_{-i\pi/2}^{i\pi/2} \prod_{-i\pi/2}^{\infty} \left[ 1 - x + s, x - s; a + s, 1 - b - s, -s \right] ds \\
= \prod_{-i\pi/2}^{\infty} \left[ 1 - x, x \right] \Phi_1[a; b; -q^{(n+1)/2 + x + b - 1}] \\
+ \prod_{-i\pi/2}^{\infty} \left[ 2 - b - x, x - 1 + b \right] \\
\times \Phi_1[1 + a - b; 2 - b; -q^{(n+1)/2 + x}] \\
\sim \prod_{-i\pi/2}^{\infty} \left[ x + a, 1 - x - a \right] \Phi_0[1 + a - b, a; ; q^{1-x-a}] \\
(2.3)
\]

and, if \( A = 1 = C = D, B = 0, \) then

\[
\frac{1}{2\pi i} \int_{-i\pi/2}^{i\pi/2} \prod_{-i\pi/2}^{\infty} \left[ b + s, 1 - x + s, x - s; a + s, -s \right] ds \\
= \prod_{-i\pi/2}^{\infty} \left[ b, 1 - x, x \right] \Phi_1[a; b; q^x] \\
\sim \prod_{-i\pi/2}^{\infty} \left[ b - a, 1 - x - a, x + a \right] \\
\times \Phi_0[a, 1 + a - b; ; -q^{1+b-2a-x-(n+1)/2}] \\
(2.4)
\]

But

\[
\Phi_1[a; b; q^{(n+1)/2 + x}] \\
= \prod \left[ 1 + a - b + x + \pi i/l \right] \Phi_1[b - a; b; -q^{1+a+x-b}] \\
(2.5)
\]

and

\[
\Phi_1[a; b; q^x] = 1/\prod [x] \Phi_1[b - a; b; -q^{(n+1)/2 + x + a - 1}] \\
(2.6)
\]
Hence we have

\[ 1\Phi_1[a; b; q^{(n+1)/2+x}] \]

\[
\sim \prod_{n=0}^{\infty} \left[ \frac{1 + a - b + x + \pi i/t, b - a - x + \pi i/t, 1 - b}{1 + a - b, b + \pi i/t - x, 1 - b + x + \pi i/t} \right] \\
\times \prod_{n=0}^{\infty} \left[ x + 1 + \pi i/t, \pi i/t - x, a \\
+ \pi \right] \\
\times 2\Phi_0[1 - a, b - a; q^{a-x-(n+1)/2}] \\
(2.7)
\]

and

\[ 1\Phi_1[a; b; q^x] \]

\[
\sim \prod_{n=0}^{\infty} \left[ \frac{2a - 1 - x, 1 - b}{1 - a, 1 + b - a - x, a - b + x} \right] \\
\times 2\Phi_0[1 - a, b - a; q^{1-x}] \\
+ \prod_{n=0}^{\infty} \left[ a + x, 1 - a - x, b - a \right] \\
\times 2\Phi_0[1 + a - b, a; -q^{1-2a+b-x-(n+1)/2}] \\
(2.8)
\]

These two results are the analogues for basic series of (1.4) above.

**References**


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