INTEGRALS FOR ASYMPTOTIC EXPANSIONS OF HYPERGEOMETRIC FUNCTIONS

L. J. SLATER

1. The integral for ordinary hypergeometric functions. In this paper I discuss integrals which provide explicit asymptotic expansions of generalized basic hypergeometric functions. The problem of asymptotic expansions for hypergeometric functions has been considered previously by C. S. Maijer [2], E. M. Wright [5], and, for basic functions, by G. N. Watson [4].

Let

\[ _{A}F_B \left[ (a) ; (b) ; z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \cdots (a_A)_n z^n}{(b_1)_n(b_2)_n \cdots (b_B)_n n!} \]

where \((a)_n = a(a+1)(a+2) \cdots (a+n-1)\). Also let

\[ \Gamma \left[ (a) ; (b) \right] = \frac{\Gamma(a_1) \Gamma(a_2) \cdots \Gamma(a_A)}{\Gamma(b_1) \Gamma(b_2) \cdots \Gamma(b_B)} \]

A dash will denote the omission of a vanishing factor in a sequence. Thus, \((a)' - a_r\) denotes the sequence \(a_1 - a_r, a_2 - a_r, \ldots, a_{r-1} - a_r, a_{r+1} - a_r, \ldots, a_A - a_r\).

It is known already (see [2] and [5]) that if

\[ I(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} z^s ds, \]

\[ \sum A(z) = \sum_{\mu=1}^{A} z^{-a_\mu} \Gamma \left[ (a)' - a_\mu, (b) + a_\mu \right] \times B_{A}F_{A+B+D-1} \left[ \begin{array}{c} (b) + a_\mu, 1 + a_\mu - (c); \\ 1 + a_\mu - (a), (d) + a_\mu; \end{array} \right] (-1)^{A+B+c+D} \]

\[ \sum B(z) = \sum_{\nu=1}^{B} z^{b_\nu} \Gamma \left[ (a) + b_\nu, (b)' - b_\nu \right] \times A_{B}F_{A+B+D} \left[ \begin{array}{c} (a) + b_\nu, 1 + b_\nu - (d); \\ 1 + b_\nu - (b), (c) + b_\nu; \end{array} \right] (-1)^{A+B+D} \]

then, provided that \(\pi |A + B - C - D| /2 > |\arg z|\),

\[ (1.1) \quad I(z) = \sum A(z) \sim \sum B(z) \quad \text{when } B + C < A + D, \]

\[ (1.2) \quad I(z) = \sum B(z) \sim \sum A(z) \quad \text{when } B + C > A + D, \]

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and (iii) \( I(1) = \sum A(1) = \sum B(1) \), when \( A - C = B - D \geq 0 \) and

\[
(1.3) \quad R \sum (c + d - a - b) > 0.
\]

In particular, the cases \( A = 1, B = 2, C = D = 0 \), and \( A = B = C = 1, D = 0 \), lead to

\[
\begin{align*}
1F1[a; b; z] & \sim \Gamma[1 + a - b; 1 - b]z^{a-1}F_0[a, 1 + a - b; ; -1/z] \\
& + e^z a^{-b} \Gamma[b; a]zF_0[1 - a, b - a; ; 1/z]
\end{align*}
\]

provided that \( |\arg z| < \pi/2 \). This is Barnes’ well-known result for the confluent hypergeometric function (see [1]).

2. The analogue for basic functions. I shall now state the corresponding results for basic hypergeometric functions together with an outline of the proof. In the usual notation for basic series, let

\[
\Phi_B[(a); (b); z] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \cdots (a_A)_n}{(b_1)_n(b_2)_n \cdots (b_B)_n(1)_n} z^n
\]

where \( (a)_n = (1 - q^a)(1 - q^{a+1}) \cdots (1 - q^{a+n-1}) \), and \( |q| < 1 \). Also let

\[
\prod_{n=0}^{p} [(a); (b)] = \prod_{n=0}^{p} \frac{(1 - q^{a+n})(1 - q^{a+n}) \cdots (1 - q^{a+n})}{(1 - q^{b+n})(1 - q^{b+n}) \cdots (1 - q^{b+n})}.
\]

Let \( I_{P,R} \) be the integral

\[
\frac{1}{2\pi i} \int \prod [(a) + s, (b) - s, 1 - z + s, z - s; (c) + s, (d) - s]ds = \int \prod (s)ds
\]

taken round the contour \( A(-i\pi/t)B(i\pi/t)C(R+i\pi/t)D(R-i\pi/t) \), and let \( I_{P,R'} \) be the same integral taken round the contour \( A(-i\pi/t) \cdot B(i\pi/t)E(-R'+i\pi/t)F(-R'-i\pi/t) \). We shall assume now that \( P > R \) and \( P > R' \), and that both contours are indented so that (supposing that \( R \) and \( R' \) are integers) the first \( R \) of every ascending sequence of poles of \( \prod P(s) \) fall inside ABCD, and the first \( R' \) of every descending sequence of poles of \( \prod P(s) \) fall inside ABED. We assume also that \( q = e^{-t}, t > 0 \), though the restriction that \( q \) is real can easily be removed from the final result by analytic continuation over the circle \( |q| < 1 \).

By the periodicity of the integrand, we have

\[
\int_{BC} \prod (s)ds = \int_{AD} \prod (s)ds \quad \text{and} \quad \int_{FA} \prod (s)ds = \int_{EB} \prod (s)ds.
\]
Hence

\[ I_{P,R} = \int_{AB} \prod_{s=1}^{P} (s)ds + \int_{CD} \prod_{s=1}^{P} (s)ds, \]

and

\[ -I_{P,R'} = \int_{AB} \prod_{s=1}^{P} (s)ds + \int_{BF} \prod_{s=1}^{P} (s)ds. \]

But

\[
I_{P,R} = \sum \text{(residues within ABCD of } \prod_{s=1}^{P} (s)),
\]

\[
= \frac{1}{l} \sum_{\mu=1}^{D} \prod_{s=1}^{P} \left[ (a) + d_{\mu}, (b) - d_{\mu}, 1 - z + d_{\mu}, z - d_{\mu}, (c) + d_{\mu}, (d') - d_{\mu}, 1 \right]
\times \sum_{n=0}^{R} \frac{((c) + d_{\mu})_n(1 + d_{\mu} - (b))_nQ^n}{((a) + d_{\mu})_n(1 + d_{\mu} - (d))_n},
\]

where \( Q = (-q^{(n+1)/2+d_{\mu}}(D-B)q_{a_1+\cdots+a_B-\delta_1-\cdots-\delta_D+z}, \)
that is

\[ I_{P,R} = \sum \prod_{s=1}^{P} \sum_{n=0}^{R} (d), \text{ say.} \]

Similarly,

\[ -I_{P,R'} = \frac{1}{l} \sum_{\rho=1}^{C} \prod_{s=1}^{P} \left[ (b) + c_{\rho}, (a) - c_{\rho}, z + c_{\rho}, 1 - z - c_{\rho}, (d) + c_{\rho}, (c') - c_{\rho}, 1 \right]
\times \sum_{n=0}^{R'} \frac{((d) + c_{\rho})_n(1 + c_{\rho} - (a))_nQ'^n}{((b) + c_{\rho})_n(1 + c_{\rho} - (c))_n},
\]

where \( Q' = (-q^{(n+1)/2+c_{\rho}}(C-A)q_{a_1+\cdots+a_A-c_1-\cdots-c_0+1-z}, \)
that is,

\[ -I_{P,R'} = \sum_{\rho=1}^{C} \prod_{s=1}^{P} \sum_{n=0}^{R'} (c), \text{ say,} \]

and so

\[
\int_{AB} \prod_{s=1}^{P} (s)ds = \sum \prod_{s=1}^{P} \sum_{n=0}^{R} (d) + \int_{DC} \prod_{s=1}^{P} (s)ds
\]

\[
= \sum \prod_{s=1}^{P} \sum_{n=0}^{R'} (c) + \int_{FB} \prod_{s=1}^{P} (s)ds.
\]

Now

\[
\left| \int_{DC} \prod_{s=1}^{P} (s)ds \right| \leq \frac{1}{2\pi} \int_{-\pi/\ell}^{\pi/\ell} \left| \prod_{s=1}^{P} (R + it) \right| dt,
\]
and
\[ \int_{\mathcal{P}B} \prod_{s} (s) ds \leq \frac{1}{2\pi} \int_{-r/1}^{r/1} \prod_{s} (-R' + ir) \ dr. \]

It has been shown previously (Slater [3]) that if \( D = B \) and \( \Re \sum (b-d) > 0 \), or if \( D > B \), \( \int_{AB} \prod_{s} (s) ds = \sum_{D} \prod_{s} \sum_{(d)} \). Also if \( A = C \) and \( \Re \sum (a-c) > 0 \), or if \( C > A \),
\[ \int_{AB} \prod_{s} (s) ds = \sum_{C} \prod_{s} \sum_{(c)}. \]

In all cases, even when \( C < A \), or when \( D < B \), we have for \( R \) fixed
\[ \int_{\mathcal{P}C} \prod_{s} (s) ds \leq \frac{1}{t} \prod_{s} \left| \frac{1 + q(s) + R + n}{1 + q^{-(s) + R + n}} \right| \left| \frac{1 + q^{-1} + R + n}{1 + q^{-1} + R + n} \right| \left| \frac{1 + q^{-c} + R + n}{1 + q^{-c} + R + n} \right| \left| \frac{1 + q^{-d} + R + n}{1 + q^{-d} + R + n} \right|. \]

But the next term of the series \( \sum_{D} \prod_{s} \sum_{R} (d) \) would be
\[ \sum_{\mu=1}^{D} \prod_{s} \left[ \frac{(c) + d_{\mu}, 1 - z + d_{\mu}, (b) - d_{\mu}, z - d_{\mu}}{(c) + d_{\mu}, (d) - d_{\mu}, 1} \right] \]
\[ \times \frac{((c) + d_{\mu})R+1(1 + d_{\mu} - (b))Q_{R+1}}{((a) + d_{\mu})R+1(1 + d_{\mu} - (d))Q_{R+1}} \]
which is of the same order in \( R \) as \( \int_{\mathcal{P}C} \prod_{s} (s) ds \). Similarly, for \( R' \) fixed \( \int_{\mathcal{P}B} \prod_{s} (s) ds \) is also bounded above as \( P \to \infty \), and this integral is of the same order in \( R' \) as the \((R'+1)\)th term of the series \( \sum_{c} \prod_{s} \sum_{R'}(c) \).

Hence we have
\[ \frac{1}{2\pi i} \int_{-r/1}^{r/1} \prod_{s} \left[ (a) + s, 1 - z + s, (b) - s, z - s; (c) + s, (d) - s \right] ds \]
\[ \sim \frac{1}{t} \sum_{\mu=1}^{D} \prod_{s} \left[ \frac{(a) + d_{\mu}, 1 - z + d_{\mu}, (b) - d_{\mu}, z - d_{\mu}}{(c) + d_{\mu}, (d) - d_{\mu}, 1} \right] \]
\[ \times_{B+C} \Phi_{A+D-1} \left[ \frac{(c) + d_{\mu}, 1 + d_{\mu} - (b)}{(a) + d_{\mu}, 1 + d_{\mu} - (d)}; Q \right] \]
\[ \times_{A+D} \Phi_{B+C-1} \left[ \frac{(d) + c_{r}, (a) - c_{r}, z + c_{r}, 1 - z - c_{r}}{(d) + c_{r}, (c) - c_{r}, 1} \right] \]
\[ \times (2.2) \]
where
\[ Q = (-q^{(n+1)/2+d_\mu})(D-B)q^{b_1+\cdots+b_d-d_1-\cdots-d_D+\varepsilon}, \]
and
\[ Q' = (-q^{(n+1)/2+e_\mu})(C-A)q^{a_1+\cdots+a_d-c_1-\cdots-c_D+1-\varepsilon}, \]
even when \( D < B \) or when \( C < A \).

In particular, let
\[ A = B = 0, \quad C = 1, \quad D = 2, \]
then
\[
\frac{1}{2\pi i} \int_{-i\pi/t}^{i\pi/t} \prod_{a=1}^{\infty} \left[ 1 - x + s, x - s; a + s, 1 - b - s, -s \right] ds
\]
\[ = \prod_{a=1}^{\infty} \left[ \begin{array}{c} 1 - x, x \\ a, 1 - b, 1 \end{array} \right] \Phi_1[a; b; -q^{(n+1)/2+x+b-1}] 
+ \prod_{a=1}^{\infty} \left[ \begin{array}{c} 2 - b - x, x - 1 + b \\ 1 + a - b, b - 1, 1 \end{array} \right] 
\times \Phi_1[1 + a - b; 2 - b; -q^{(n+1)/2+x}] 
\]
\[ \sim \prod_{a=1}^{\infty} \left[ \begin{array}{c} x + a, 1 - x - a \\ 1 + a - b, a, 1 \end{array} \right] \Phi_0[1 + a - b, a; q^{1-x-a}] \]
and, if \( A = C = D, B = 0 \), then
\[
\frac{1}{2\pi i} \int_{-i\pi/t}^{i\pi/t} \prod_{a=1}^{\infty} \left[ b + s, 1 - x + s, x - s; a + s, -s \right] ds
\]
\[ = \prod_{a=1}^{\infty} \left[ b, 1 - x, x \\ a, 1 \right] \Phi_1[a; b; q^x] 
\]
\[ \sim \prod_{a=1}^{\infty} \left[ b - a, 1 - x - a, x + a \\ a, 1 \right] 
\times \Phi_0[a, 1 + a - b; -q^{1+b-2a-x-(n+1)/2}] . \]

But
\[ \Phi_1[a; b; q^{(n+1)/2+x}] \]
\[ = \prod_{a=1}^{\infty} \left[ 1 + a - b + x + \pi i/t \right] \Phi_1[b - a; b; -q^{1+a+x-b}] \]
and
\[ \Phi_1[a; b; q^x] = 1/\prod_{a=1}^{\infty} \left[ x \right] \Phi_1[b - a; b; -q^{(n+1)/2+x+a-1}]. \]
Hence we have

\[ 1\Phi_1[a; b; q^{(n+1)/2+x}] \]
\[ \sim \prod_{i=0}^{\infty} \left[ \begin{array}{l} 1 + a - b + x + \pi i/t, b - a - x + \pi i/t, 1 - b \\ 1 + a - b, b + \pi i/t - x, 1 - b + x + \pi i/t \end{array} \right] \]
\[ \times 2\Phi_0[a, 1 + a - b; -q^{b-a-x}] \]
\[ + \prod_{i=0}^{\infty} \left[ \begin{array}{l} x + 1 + \pi i/t, \pi i/t - x, a \\ b, 1 + a - b + x + \pi i/t, b - a - x + \pi i/t \end{array} \right] \]
\[ \times 2\Phi_0[1 - a, b - a; q^{a-x-(n+1)/2}] \]

(2.7)

and

\[ 1\Phi_1[a; b; q^x] \]
\[ \sim \prod_{i=0}^{\infty} \left[ \begin{array}{l} 2a - 1 - x, 1 - b \\ 1 - a, 1 + b - a - x, a - b + x \end{array} \right] \]
\[ \times 2\Phi_0[1 - a, b - a; q^{1-x}] \]
\[ + \prod_{i=0}^{\infty} \left[ \begin{array}{l} a + x, 1 - a - x, b - a \\ x, b, x, 1 - x \end{array} \right] \]
\[ \times 2\Phi_0[1 + a - b, a; -q^{1-2a+b-x-(n+1)/2}] \]

(2.8)

These two results are the analogues for basic series of (1.4) above.

**REFERENCES**


**Newnham College, Cambridge University**