INTEGRALS FOR ASYMPTOTIC EXPANSIONS OF HYPERGEOMETRIC FUNCTIONS

L. J. SLATER

1. **The integral for ordinary hypergeometric functions.** In this paper I discuss integrals which provide explicit asymptotic expansions of generalized basic hypergeometric functions. The problem of asymptotic expansions for hypergeometric functions has been considered previously by C. S. Maijer [2], E. M. Wright [5], and, for basic functions, by G. N. Watson [4].

Let

$$ {}_AF_B[(a);(b);z] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \cdots (a_A)_n z^n}{(b_1)_n(b_2)_n \cdots (b_B)_n n!} $$

where $(a)_n = a(a+1)(a+2) \cdots (a+n-1)$. Also let

$$ \Gamma[(a);(b)] = \frac{\Gamma(a_1)\Gamma(a_2) \cdots \Gamma(a_A)}{\Gamma(b_1)\Gamma(b_2) \cdots \Gamma(b_B)}. $$

A dash will denote the omission of a vanishing factor in a sequence. Thus, $(a)'-a_r$ denotes the sequence a_1-a_r, a_2-a_r, $\cdots$, $a_{r-1}-a_r$, $a_{r+1}-a_r$, $\cdots$, a_A-a_r.

It is known already (see [2] and [5]) that *if*

$$ I(z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma\begin{bmatrix}(a)+s, (b)-s\\ (c)+s, (d)-s\end{bmatrix} z^s ds, $$

$$ \sum{}_A(z) = \sum_{\mu=1}^{A} z^{-a_\mu}\Gamma\begin{bmatrix}(a)'-a_\mu, (b)+a_\mu\\ (c)-a_\mu, (d)+a_\mu\end{bmatrix} $$

$$ \times {}_{B+C}F_{A+D-1}\begin{bmatrix}(b)+a_\mu, 1+a_\mu-(c);\\ 1+a_\mu-(a), (d)+a_\mu;\end{bmatrix} (-1)^{A+C}z^{-1}\Bigg], $$

$$ \sum{}_B(z) = \sum_{\nu=1}^{B} z^{b_\nu}\Gamma\begin{bmatrix}(a)+b_\nu, (b)'-b_\nu\\ (c)+b_\nu, (d)-b_\nu\end{bmatrix} $$

$$ \times {}_{A+D}F_{B+C-1}\begin{bmatrix}(a)+b_\nu, 1+b_\nu-(d);\\ 1+b_\nu-(b), (c)+b_\nu;\end{bmatrix} (-1)^{B+D}z\Bigg] $$

then, provided that $\pi|A+B-C-D|/2 > |\arg z|$,

(1.1) (i) $I(z) = \sum{}_A(z) \sim \sum{}_B(z)$ *when* $B+C<A+D$,

(1.2) (ii) $I(z) = \sum{}_B(z) \sim \sum{}_A(z)$ *when* $B+C>A+D$,

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and (iii) $I(1)=\sum_A(1)=\sum_B(1)$, *when* $A-C=B-D\geqq 0$ *and*

$$R1\sum(c+d-a-b)>0. \tag{1.3}$$

In particular, the cases $A=1$, $B=2$, $C=D=0$, and $A=B=C=1$, $D=0$, lead to

$$\begin{aligned} {}_1F_1[a;b;z] \sim{}& \Gamma[1+a-b;1-b]z^{-a}{}_2F_0[a,1+a-b;\ \ ;-1/z] \\ &+e^z z^{a-b}\Gamma[b;a]{}_2F_0[1-a,b-a;\ \ ;1/z] \end{aligned} \tag{1.4}$$

provided that $|\arg z|<\pi/2$. This is Barnes' well-known result for the confluent hypergeometric function (see [1]).

2. **The analogue for basic functions.** I shall now state the corresponding results for basic hypergeometric functions together with an outline of the proof. In the usual notation for basic series, let

$${}_A\Phi_B[(a);(b);z]=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_A)_n z^n}{(b_1)_n(b_2)_n\cdots(b_B)_n(1)_n}$$

where $(a)_n=(1-q^a)(1-q^{a+1})\cdots(1-q^{a+n-1})$, and $|q|<1$. Also let

$$\prod^P[(a);(b)]=\prod_{n=0}^{P}\frac{(1-q^{a_1+n})(1-q^{a_2+n})\cdots(1-q^{a_A+n})}{(1-q^{b_1+n})(1-q^{b_2+n})\cdots(1-q^{b_B+n})}.$$

Let $I_{P,R}$ be the integral

$$\frac{1}{2\pi i}\int\prod^P[(a)+s,(b)-s,1-z+s,z-s;(c)+s,(d)-s]ds$$
$$=\int\prod^P(s)ds$$

taken round the contour $A(-i\pi/t)B(i\pi/t)C(R+i\pi/t)D(R-i\pi/t)$, and let $I_{P,R'}$ be the same integral taken round the contour $A(-i\pi/t)\cdot B(i\pi/t)E(-R'+i\pi/t)F(-R'-i\pi/t)$. We shall assume now that $P>R$ and $P>R'$, and that both contours are indented so that (supposing that R and R' are integers) the first R of every ascending sequence of poles of $\prod^P(s)$ fall inside $ABCD$, and the first R' of every descending sequence of poles of $\prod^P(s)$ fall inside $ABEF$. We assume also that $q=e^{-t}$, $t>0$, though the restriction that q is real can easily be removed from the final result by analytic continuation over the circle $|q|<1$.

By the periodicity of the integrand, we have

$$\int_{BC}\prod^P(s)ds=\int_{AD}\prod^P(s)ds \quad\text{and}\quad \int_{FA}\prod^P(s)ds=\int_{EB}\prod^P(s)ds.$$

Hence

$$I_{P,R} = \int_{AB} \prod^{P}(s)ds + \int_{CD} \prod^{P}(s)ds,$$

and

$$-I_{P,R'} = \int_{AB} \prod^{P}(s)ds + \int_{EF} \prod^{P}(s)ds.$$

But

$$\begin{aligned} I_{P,R} &= \sum (\text{residues within } ABCD \text{ of } \prod^{P}(s)), \\ &= \frac{1}{t}\sum_{\mu=1}^{D}\prod^{P}\begin{bmatrix}(a)+d_\mu,\ (b)-d_\mu,\ 1-z+d_\mu,\ z-d_\mu \\ (c)+d_\mu,\ (d)'-d_\mu,\ 1\end{bmatrix} \\ &\quad \times \sum_{n=0}^{R}\frac{((c)+d_\mu)_n(1+d_\mu-(b))_n Q^n}{((a)+d_\mu)_n(1+d_\mu-(d))_n} \end{aligned}$$

where $Q=(-q^{(n+1)/2+d_\mu})^{(D-B)}q^{b_1+\cdots+b_B-d_1-\cdots-d_D+z}$, that is

$$I_{P,R} = \sum^{D}\prod^{P}\sum^{R}(d), \quad \text{say.}$$

Similarly,

$$\begin{aligned} -I_{P,R'} &= \frac{1}{t}\sum_{\nu=1}^{C}\prod^{P}\begin{bmatrix}(b)+c_\nu,\ (a)-c_\nu,\ z+c_\nu,\ 1-z-c_\nu \\ (d)+c_\nu,\ (c)'-c_\nu,\ 1\end{bmatrix} \\ &\quad \times \sum_{n=0}^{R'}\frac{((d)+c_\nu)_n(1+c_\nu-(a))_n Q'^n}{((b)+c_\nu)_n(1+c_\nu-(c))_n} \end{aligned}$$

where $Q'=(-q^{(n+1)/2+c_\nu})^{(C-A)}q^{a_1+\cdots+a_A-c_1-\cdots-c_C+1-z}$, that is,

$$-I_{P,R'} = \sum^{C}\prod^{P}\sum^{R'}(c), \quad \text{say,}$$

and so

$$\begin{aligned} \int_{AB}\prod^{P}(s)ds &= \sum^{D}\prod^{P}\sum^{R}(d) + \int_{DC}\prod^{P}(s)ds \\ &= \sum^{C}\prod^{P}\sum^{R'}(c) + \int_{FE}\prod^{P}(s)ds. \end{aligned} \tag{2.1}$$

Now

$$\left|\int_{DC}\prod^{P}(s)ds\right| \leqq \frac{1}{2\pi}\int_{-\pi/t}^{\pi/t}\left|\prod^{P}(R+ir)\right|dr,$$

and

$$\left|\int_{FE}\prod^{P}(s)ds\right| \leqq \frac{1}{2\pi}\int_{-\pi/t}^{\pi/t}\left|\prod^{P}(-R'+ir)\right|dr.$$

It has been shown previously (Slater [3]) that if $D=B$ and $\Re\sum(b-d)>0$, or if $D>B$, $\int_{AB}\prod^{P}(s)ds=\sum^{D}\prod^{\infty}\sum^{\infty}(d)$. Also if $A=C$ and $\Re\sum(a-c)>0$, or if $C>A$,

$$\int_{AB}\prod^{P}(s)ds = \sum^{C}\prod^{\infty}\sum^{\infty}(c).$$

In all cases, even when $C<A$, or when $D<B$, we have for R fixed

$$\left|\int_{DC}\prod^{P}(s)ds\right|$$
$$\leqq\frac{1}{t}\prod^{\infty}\frac{|1+q^{(a)+R+n}|\,|1+q^{1-z+R+n}|\,|1+q^{(b)-R+n}|\,|1+q^{z-R+n}|}{|1-q^{(c)+R+n}|\,|1-q^{(d)-R+n}|}.$$

But the next term of the series $\sum^{D}\prod^{P}\sum^{R}(d)$ would be

$$\sum_{\mu=1}^{D}\prod^{P}\begin{bmatrix}(a)+d_\mu,\ 1-z+d_\mu,\ (b)-d_\mu,\ z-d_\mu\\ (c)+d_\mu,\ (d)'-d_\mu,\ 1\end{bmatrix}$$
$$\times\frac{((c)+d_\mu)_{R+1}(1+d_\mu-(b))_{R+1}Q^{R+1}}{((a)+d_\mu)_{R+1}(1+d_\mu-(d))_{R+1}}$$

which is of the same order in R as $\int_{DC}\prod^{P}(s)ds$. Similarly, for R' fixed $\int_{FE}\prod^{P}(s)ds$ is also bounded above as $P\to\infty$, and this integral is of the same order in R' as the $(R'+1)$th term of the series $\sum^{C}\prod^{P}\sum^{R'}(c)$.

Hence we have

$$\frac{1}{2\pi i}\int_{-i\pi/t}^{i\pi/t}\prod^{\infty}[(a)+s,1-z+s,(b)-s,z-s;(c)+s,(d)-s]ds$$
$$\sim\frac{1}{t}\sum_{\mu=1}^{D}\prod^{\infty}\begin{bmatrix}(a)+d_\mu,\ 1-z+d_\mu,\ (b)-d_\mu,\ z-d_\mu\\ (c)+d_\mu,\ (d)'-d_\mu,\ 1\end{bmatrix}$$
$$\times{}_{B+C}\Phi_{A+D-1}\begin{bmatrix}(c)+d_\mu,\ 1+d_\mu-(b);\\ (a)+d_\mu,\ 1+d_\mu-(d)';\end{bmatrix}Q$$
$$(2.2)\qquad\sim\frac{1}{t}\sum_{\nu=1}^{C}\prod^{\infty}\begin{bmatrix}(b)+c_\nu,\ (a)-c_\nu,\ z+c_\nu,\ 1-z-c_\nu\\ (d)+c_\nu,\ (c)'-c_\nu,\ 1\end{bmatrix}$$
$$\times{}_{A+D}\Phi_{B+C-1}\begin{bmatrix}(d)+c_\nu,\ 1+c_\nu-(a);\\ (b)+c_\nu,\ 1+c_\nu-(c)';\end{bmatrix}Q'$$

where

$$Q = (-q^{(n+1)/2+d_\mu})^{(D-B)} q^{b_1+\cdots+b_B-d_1-\cdots-d_D+z},$$

and

$$Q' = (-q^{(n+1)/2+c_\nu})^{(C-A)} q^{a_1+\cdots+a_A-c_1-\cdots-c_C+1-z},$$

even when $D<B$ or when $C<A$.

In particular, let

$$A = B = 0, \qquad C = 1, \qquad D = 2,$$

then

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-i\pi/t}^{i\pi/t} \prod^{\infty} [1-x+s,\ x-s;\ a+s,\ 1-b-s,\ -s]ds \\
&\quad = \prod^{\infty}\begin{bmatrix} 1-x,\ x \\ a,\ 1-b,\ 1 \end{bmatrix} {}_1\Phi_1[a;\ b;\ -q^{(n+1)/2+x+b-1}] \\
&\qquad + \prod^{\infty}\begin{bmatrix} 2-b-x,\ x-1+b \\ 1+a-b,\ b-1,\ 1 \end{bmatrix} \\
&\qquad \times {}_1\Phi_1[1+a-b;\ 2-b;\ -q^{(n+1)/2+x}] \\
&\quad \sim \prod^{\infty}\begin{bmatrix} x+a,\ 1-x-a \\ 1+a-b,\ a,\ 1 \end{bmatrix} {}_2\Phi_0[1+a-b,\ a;\ \ ;\ q^{1-x-a}]
\end{aligned} \tag{2.3}$$

and, if $A=1=C=D$, $B=0$, then

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-i\pi/t}^{i\pi/t} \prod^{\infty} [b+s,\ 1-x+s,\ x-s;\ a+s,\ -s]ds \\
&\quad = \prod^{\infty}\begin{bmatrix} b,\ 1-x,\ x \\ a,\ 1 \end{bmatrix} {}_1\Phi_1[a;\ b;\ q^x] \\
&\quad \sim \prod^{\infty}\begin{bmatrix} b-a,\ 1-x-a,\ x+a \\ a,\ 1 \end{bmatrix} \\
&\qquad \times {}_2\Phi_0[a,\ 1+a-b;\ \ ;\ -q^{1+b-2a-x-(n+1)/2}].
\end{aligned} \tag{2.4}$$

But

$$\begin{aligned}
&{}_1\Phi_1[a;\ b;\ q^{(n+1)/2+x}] \\
&\quad = \prod^{\infty} [1+a-b+x+\pi i/t]\,{}_1\Phi_1[b-a;\ b;\ -q^{1+a+x-b}]
\end{aligned} \tag{2.5}$$

and

$${}_1\Phi_1[a;\ b;\ q^x] = 1\Big/\prod^{\infty} [x]\cdot {}_1\Phi_1[b-a;\ b;\ -q^{(n+1)/2+x+a-1}]. \tag{2.6}$$

Hence we have

$$
\begin{aligned}
&{}_1\Phi_1[a;\, b;\, q^{(n+1)/2+x}] \\
&\quad \sim \prod^{\infty}\begin{bmatrix} 1+a-b+x+\pi i/t,\ b-a-x+\pi i/t,\ 1-b \\ 1+a-b,\ b+\pi i/t-x,\ 1-b+x+\pi i/t \end{bmatrix} \\
&\qquad \times {}_2\Phi_0[a,\ 1+a-b;\ \ ;\ -q^{b-a-x}] \\
&\qquad + \prod^{\infty}\begin{bmatrix} x+1+\pi i/t,\ \pi i/t-x,\ a \\ b,\ 1+a-b+x+\pi i/t,\ b-a-x+\pi i/t \end{bmatrix} \\
&\qquad \times {}_2\Phi_0[1-a,\ b-a;\ \ ;\ q^{a-x-(n+1)/2}]
\end{aligned}
\tag{2.7}
$$

and

$$
\begin{aligned}
&{}_1\Phi_1[a;\, b;\, q^{x}] \\
&\quad \sim \prod^{\infty}\begin{bmatrix} 2a-1-x,\ 1-b \\ 1-a,\ 1+b-a-x,\ a-b+x \end{bmatrix} \\
&\qquad \times {}_2\Phi_0[1-a,\ b-a;\ \ ;\ q^{1-x}] \\
&\qquad + \prod^{\infty}\begin{bmatrix} a+x,\ 1-a-x,\ b-a \\ x,\ b,\ x,\ 1-x \end{bmatrix} \\
&\qquad \times {}_2\Phi_0[1+a-b,\ a;\ \ ;\ -q^{1-2a+b-x-(n+1)/2}].
\end{aligned}
\tag{2.8}
$$

These two results are the analogues for basic series of (1.4) above.

References

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Newnham College, Cambridge University