A BASIC SET OF HOMOGENEOUS HARMONIC POLYNOMIALS IN k VARIABLES

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1. The authors present basic sets of:
   (a) homogeneous harmonic polynomials of degree \( n \) in \( k \) variables, \( k \geq 3 \);
   (b) associated polynomial solutions of the wave equation, and
   (c) analogous solutions for \( \sum_{j=1}^k \left( \partial^s u / \partial x_j^s \right) = 0, s = 3, 4, \ldots \).

2. For any set of non-negative integers \( (b_j) \) such that \( b_1 \leq 1 \) and \( \sum_{j=1}^k b_j = n \), let

\[
H^n_{b_1 b_2 \cdots b_k}(x_1, x_2, \ldots, x_k)
\]

\[
= \sum (-1)^{[a_1/2]} \frac{n!}{\prod_{j=1}^k a_j!} \frac{\lfloor a_1 \rfloor!}{\prod_{j=2}^k \left( \frac{b_j - a_j}{2} \right)!} \prod_{j=1}^k x_j^{a_j}
\]

where the summation is extended over all \( (a_j) \) such that:

1. \( a_j \equiv b_j \mod 2, j = 1, 2, \ldots, k, \)
2. \( \sum_{j=1}^k a_j = n, \)
3. \( a_j \leq b_j, j = 2, 3, \ldots, k. \)

The polynomials (1) form a basic set of homogeneous harmonic polynomials in \( k \) variables. The proof is given in three parts.

A. The polynomials (1) are linearly independent since each contains exactly one different nonvanishing term of the monomials \( x_1^{a_1}x_2^{a_2} \cdots x_k^{a_k}, \sum_{j=1}^k a_j = n, a_1 \leq 1. \)

Moreover, since the number of terms in \( (\sum_{j=1}^k x_j)^n \) is

\[
\binom{n + k - 1}{k - 1},
\]

the total number of monomials of type \( x_2^{a_2}x_3^{a_3} \cdots x_k^{a_k}, \sum_{j=2}^k a_j = n, \) and of type \( x_1x_2^{a_2}x_3^{a_3} \cdots x_k^{a_k}, \sum_{j=2}^k a_j = n - 1, \) is

\[
\binom{n + k - 2}{k - 2} + \binom{n + k - 3}{k - 2} = \binom{n + k - 3}{k - 3} \left( \frac{2n}{k - 2} + 1 \right).
\]

Thus the polynomials (1) are

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\[ \binom{n + k - 3}{k - 3} \left( \frac{2n}{k - 2} + 1 \right) \]

in number.

B. They are harmonic. Let \( c_j, j = 1, \cdots, k \), be such that

1. \( c_j \equiv b \mod 2 \),
2. \( c_j \leq b_j, j = 2, 3, \cdots, k \), and
3. \( \sum_{j=1}^{k} c_j = n - 2 \).

The coefficient \( B_{c_1, \cdots, c_k} \) of \( \prod_{j=1}^{k} x_j^{c_j} \) in \( \nabla^2 H_{b_1, \cdots, b_k} (x_1, \cdots, x_k) \) is given by

\[
B_{c_1, \cdots, c_k} = (-1)^{\left\lfloor c_1/2 \right\rfloor + 1} \frac{n!}{\prod_{j=1}^{k} c_j!} \cdot \left[ \sum_{j=2}^{k} \frac{c_1}{2} \left( \frac{b_j - c_i}{2} \right) ! \right] - \frac{n! \left\lfloor \frac{c_1}{2} \right\rfloor !}{\prod_{j=1}^{k} \left( \frac{b_j - c_i}{2} \right) !} \cdot \left[ \sum_{j=2}^{k} \frac{b_j}{2} + \frac{1}{2} \sum_{j=2}^{k} c_j \right] \]

\[
= (-1)^{\left\lfloor c_1/2 \right\rfloor + 1} \frac{n! \left\lfloor \frac{c_1}{2} \right\rfloor !}{\prod_{j=1}^{k} c_j! \prod_{j=2}^{k} \left( \frac{b_j - c_i}{2} \right) !} \cdot \left[ \frac{c_1}{2} + 1 - \frac{1}{2} \sum_{j=2}^{k} b_j + \frac{1}{2} \sum_{j=2}^{k} c_j \right] \]

\[
= (-1)^{\left\lfloor c_1/2 \right\rfloor + 1} \frac{n! \left\lfloor \frac{c_1}{2} \right\rfloor !}{\prod_{j=1}^{k} c_j! \prod_{j=2}^{k} \left( \frac{b_j - c_i}{2} \right) !} \cdot \left[ \frac{c_1}{2} + 1 - \frac{1}{2} (n - b_1) + \frac{1}{2} (n - 2 - c_1) \right] \]

\[
= (-1)^{\left\lfloor c_1/2 \right\rfloor + 1} \frac{n! \left\lfloor \frac{c_1}{2} \right\rfloor !}{\prod_{j=1}^{k} c_j! \prod_{j=2}^{k} \left( \frac{b_j - c_i}{2} \right) !} \cdot \left[ \frac{c_1}{2} - \frac{1}{2} (c_1 - b_1) \right] \equiv 0. \]
C. For a general homogeneous polynomial $H^k_n$ of degree $n$ in $k$ variables the vanishing of the Laplacian $\nabla^2 H^k_n$ provides
\[
\binom{n+k-3}{k-1} \text{ equations on the } \binom{n+k-1}{k-1} \text{ coefficients}
\]
of $H^k_n$. Thus the number of linearly independent homogeneous harmonic polynomials of degree $n$ in $k$ variables is
\[
\binom{n+k-1}{k-1} - \binom{n+k-3}{k-1} = \binom{n+k-3}{k-3} \left( \frac{2n}{k-2} + 1 \right),
\]
which is the number of polynomials (1).

3. It is worth noting that the polynomials obtained from (1) by deleting the factor $(-1)^{l_1/2}$ are solutions of the generalized wave equation $\sum_{j=2}^{k} \partial^2 u/\partial x_j^2 = \partial^2 u/\partial x_1^2$, which form a basic set for that equation.

4. Further, for each set of $k$ non-negative integers $b_j$ such that
\[
\sum_{j=1}^{k} b_j = n, \quad b_j \leq s-1,
\]
the polynomials
\[
H_{b_1, b_2, \cdots, b_k}(x_1, x_2, \cdots, x_k)
\]
\[
= \sum (-1)^{[a_1/s]} \frac{n!}{\prod_{j=1}^{k} a_j!} \frac{[a_1/s]!}{\prod_{j=2}^{k} (b_j - a_j)!} \prod_{j=1}^{k} x_j^{a_j},
\]
where the summation extends over all $a_j$ such that
(1) $a_j \equiv b_j \text{ mod } s, \quad j = 1, 2, \cdots, k$,
(2) $\sum_{j=1}^{k} a_j = n$,
(3) $a_j \leq b_j, \quad j = 2, 3, \cdots, k$,
provide a basic set of solutions for
\[
\sum_{j=1}^{k} \frac{\partial^s u}{\partial x_j^s} = 0.
\]

5. Of particular interest for harmonic polynomials is the case $k = 3$. Whittaker\(^1\) has obtained the general solution of $\nabla^2 U(x, y, z) = 0$ by means of an integral. Ketchum\(^2\) gives another form of the general solution as an analytic function of a hypervariable $w$ such that the $2n+1$ linearly independent components of $w^n$ form a basic set of homogeneous harmonic polynomials of degree $n$. Both of these re-

\(^1\) Math. Ann. vol. 57 (1903) p. 333.
\(^2\) Amer. J. Math. vol. 51 (1929) p. 179.
suits use trigonometric functions, and neither of them displays immediately a set of polynomial solutions of degree \( n \). Morse and Feshbach\(^3\) indicate how one obtains, from a special case of Whittaker’s integral, a basic set of degree \( n \), but carry the computation only as far as \( n=3 \). Courant and Hilbert\(^4\) give a basic set with only one of the \( 2n+1 \) members having real coefficients. The polynomials (1) for \( k=3 \), \( x_1=x \), \( x_2=y \) and \( x_3=z \), unlike those from the basic sets referred to above, are given explicitly for each \( n \). Thus, for \( n=6 \) the 13 independent spherical harmonics are obtained by assigning \((b_1 \ b_2 \ b_3)\) the values \((0 \ 6 \ 0), (0 \ 5 \ 1), (0 \ 4 \ 2), (0 \ 3 \ 3), (0 \ 2 \ 4), (0 \ 1 \ 5), (0 \ 0 \ 6), (1 \ 5 \ 0), (1 \ 4 \ 1) (1 \ 3 \ 2), (1 \ 2 \ 3), (1 \ 1 \ 4), \) and \((1 \ 0 \ 5)\) in turn. We display a typical member,

\[
H_{123}^6 = 60 x^2 y^2 z^2 - 60 x^3 y z - 20 x^2 z^2 + 12 x z^5.
\]

6. The authors wish to express their appreciation to Professor Ernest Ikenberry who directed their attention to the 3-dimensional basic sets given by Morse-Feshbach and Courant-Hilbert and to the referee who pointed out the construction of basic sets of \( p+2 \) dimensions appearing in *Higher transcendental functions*,\(^5\) A. Erdélyi, editor. These sets for \( p+2 \) dimensions differ from those of the authors in that the coefficients are in general complex while those of the authors are real.

*Added in proof.* When these results for \( k=3 \) only were presented at the International Congress of Mathematicians, September, 1954, Professors P. C. Rosenbloom and L. Bers kindly called the authors’ attention to a three variable basic set of harmonic polynomials given by M. H. Protter [*Generalized spherical harmonics*, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 314–341]. In a forthcoming note in the Proceedings, the authors point out that their results for \( k=3 \) give a single formulation for the four classes into which Protter’s basic set was divided.

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