A BASIC SET OF HOMOGENEOUS HARMONIC POLYNOMIALS IN $k$ VARIABLES

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1. The authors present basic sets of:
   (a) homogeneous harmonic polynomials of degree $n$ in $k$ variables, $k \geq 3$;
   (b) associated polynomial solutions of the wave equation, and
   (c) analogous solutions for $\sum_{j=1}^{k} (\partial^s u)/(\partial x_j^s) = 0$, $s = 3, 4, \cdots$.

2. For any set of non-negative integers $(b_j)$ such that $b_1 \leq 1$ and $\sum_{j=1}^{k} b_j = n$, let

$$H_{b_1,b_2,\ldots,b_k}^n(x_1, x_2, \ldots, x_k)$$

where the summation is extended over all $(a_j)$ such that:

1. $a_j \equiv b_j \mod 2$, $j = 1, 2, \cdots, k$,
2. $\sum_{j=1}^{k} a_j = n$,
3. $a_j \leq b_j$, $j = 2, 3, \cdots, k$.

The polynomials (1) form a basic set of homogeneous harmonic polynomials in $k$ variables. The proof is given in three parts.

A. The polynomials (1) are linearly independent since each contains exactly one different nonvanishing term of the monomials $x_1^{a_1}x_2^{a_2} \cdots x_k^{a_k}$, $\sum_{j=1}^{k} a_j = n$, $a_1 \leq 1$.

Moreover, since the number of terms in $(\sum_{j=1}^{k} x_j)^n$ is

$$\binom{n+k-1}{k-1},$$

the total number of monomials of type $x_2^{a_2}x_3^{a_3} \cdots x_k^{a_k}$, $\sum_{j=2}^{k} a_j = n$, and of type $x_1 x_2^{a_2}x_3^{a_3} \cdots x_k^{a_k}$, $\sum_{j=2}^{k} a_j = n-1$, is

$$\binom{n+k-2}{k-2} + \binom{n+k-3}{k-2} = \binom{n+k-3}{k-3}\left(\frac{2n}{k-2} + 1\right).$$

Thus the polynomials (1) are

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B. They are harmonic. Let $c_j, j = 1, \cdots, k$, be such that

1. $c_j \equiv b \mod 2$,
2. $c_j \leq b_j, j = 2, 3, \cdots, k$, and
3. $\sum_{j=1}^{k} c_j = n - 2$.

The coefficient $B_{c_1, c_2, \ldots, c_k}$ of $\prod_{j=1}^{k} x_j^{c_j}$ in $V^2H_{b_1, \ldots, b_k}(x_1, \ldots, x_k)$ is given by

$$B_{c_1, \ldots, c_k} = (-1)^{\left\lfloor c_1/2 \right\rfloor + 1} \frac{n!}{\prod_{j=1}^{k} c_j!} \left( \prod_{j=1}^{k} \left( \frac{b_j - c_j}{2} \right)! \right) \left( \frac{c_1 + 1}{2} \right)! \sum_{j=2}^{k} \left\lfloor \frac{c_1}{2} \right\rfloor! \left( \frac{b_j - c_j}{2} \right)! - \sum_{j=2}^{k} \left\lfloor \frac{c_1}{2} \right\rfloor! \left( \frac{b_j - c_j}{2} \right)!$$

$$= (-1)^{\left\lfloor c_1/2 \right\rfloor + 1} \frac{n! \left\lfloor \frac{c_1}{2} \right\rfloor!}{\prod_{j=1}^{k} c_j! \prod_{j=2}^{k} \left( \frac{b_j - c_j}{2} \right)!} \left( \frac{c_1}{2} \right)! \left[ 1 - \frac{1}{2} \sum_{j=2}^{k} b_j + \frac{1}{2} \sum_{j=2}^{k} c_j \right]$$

$$= (-1)^{\left\lfloor c_1/2 \right\rfloor + 1} \frac{n! \left\lfloor \frac{c_1}{2} \right\rfloor!}{\prod_{j=1}^{k} c_j! \prod_{j=2}^{k} \left( \frac{b_j - c_j}{2} \right)!} \left( \frac{c_1}{2} \right)! \left[ 1 - \frac{1}{2} (n - b_1) + \frac{1}{2} (n - 2 - c_1) \right]$$

$$= (-1)^{\left\lfloor c_1/2 \right\rfloor + 1} \frac{n! \left\lfloor \frac{c_1}{2} \right\rfloor!}{\prod_{j=1}^{k} c_j! \prod_{j=2}^{k} \left( \frac{b_j - c_j}{2} \right)!} \left( \frac{c_1}{2} \right)! \left[ 1 - \frac{1}{2} (c_1 - b_1) \right] \equiv 0.$$
C. For a general homogeneous polynomial $H^a_k$ of degree $n$ in $k$ variables the vanishing of the Laplacian $\nabla^2 H^a_k$ provides
\[
\binom{n+k-3}{k-1} \text{ equations on the } \binom{n+k-1}{k-1} \text{ coefficients of } H^a_k.
\]
Thus the number of linearly independent homogeneous harmonic polynomials of degree $n$ in $k$ variables is
\[
\binom{n+k-1}{k-1} - \binom{n+k-3}{k-1} = \binom{n+k-3}{k-3} \left(\frac{2n}{k-2} + 1\right),
\]
which is the number of polynomials (1).

3. It is worth noting that the polynomials obtained from (1) by deleting the factor $(-1)^{\lfloor a_i/2 \rfloor}$ are solutions of the generalized wave equation $\sum_{i=2}^k \partial^2 u/\partial x_i^2 = \partial^2 u/\partial x_1^2$, which form a basic set for that equation.

4. Further, for each set of $k$ non-negative integers $b_j$ such that
\[
\sum_{j=1}^k b_j = n, \quad b_1 \leq s-1,
\]
the polynomials
\[
H^{n,*}_{b_1, b_2, \ldots, b_k}(x_1, x_2, \ldots, x_k)
\]
\[
= \sum (-1)^{\lfloor a_i/2 \rfloor} \frac{n!}{\prod_{j=1}^k a_j!} \frac{\left[ a_1 \right]!}{s!} \frac{\prod_{j=2}^k \left( b_j - a_j \right)!}{s!} \prod_{j=1}^k x_j^{a_j},
\]
where the summation extends over all $a_j$ such that
\begin{enumerate}
\item $a_j \equiv b_j \mod s, \; j = 1, 2, \ldots, k,$
\item $\sum_{j=1}^k a_j = n,$
\item $a_j \leq b_j, \; j = 2, 3, \ldots, k,$
\end{enumerate}
provide a basic set of solutions for
\[
\sum_{j=1}^k \frac{\partial^2 u}{\partial x_j^2} = 0.
\]

5. Of particular interest for harmonic polynomials is the case $k=3$. Whittaker\(^1\) has obtained the general solution of $\nabla^2 U(x, y, z) = 0$ by means of an integral. Ketchum\(^2\) gives another form of the general solution as an analytic function of a hypervariable $w$ such that the $2n+1$ linearly independent components of $w^n$ form a basic set of homogeneous harmonic polynomials of degree $n$. Both of these re-

\(^1\) Math. Ann. vol. 57 (1903) p. 333.
\(^2\) Amer. J. Math. vol. 51 (1929) p. 179.
suits use trigonometric functions, and neither of them displays immediately a set of polynomial solutions of degree $n$. Morse and Feshbach\(^3\) indicate how one obtains, from a special case of Whittaker’s integral, a basic set of degree $n$, but carry the computation only as far as $n=3$. Courant and Hilbert\(^4\) give a basic set with only one of the $2n+1$ members having real coefficients. The polynomials (1) for $k=3$, $x_1=x$, $x_2=y$ and $x_3=z$, unlike those from the basic sets referred to above, are given explicitly for each $n$. Thus, for $n=6$ the 13 independent spherical harmonics are obtained by assigning $(b_1 b_2 b_3)$ the values $(0 6 0)$, $(0 5 1)$, $(0 4 2)$, $(0 3 3)$, $(0 2 4)$, $(0 1 5)$, $(0 0 6)$, $(1 5 0)$, $(1 4 1)$, $(1 3 2)$, $(1 2 3)$, $(1 1 4)$, and $(1 0 5)$ in turn. We display a typical member,

$$H_{123}^6 = 60xy^2z^3 - 60xyz^2 - 20x^3z + 12x^5z.$$  

6. The authors wish to express their appreciation to Professor Ernest Ikenberry who directed their attention to the 3-dimensional basic sets given by Morse-Feshbach and Courant-Hilbert and to the referee who pointed out the construction of basic sets of $p+2$ dimensions appearing in *Higher transcendental functions*,\(^5\) A. Erdélyi, editor. These sets for $p+2$ dimensions differ from those of the authors in that the coefficients are in general complex while those of the authors are real.

*Added in proof.* When these results for $k=3$ only were presented at the International Congress of Mathematicians, September, 1954, Professors P. C. Rosenbloom and L. Bers kindly called the authors' attention to a three variable basic set of harmonic polynomials given by M. H. Protter [Generalized spherical harmonics, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 314–341]. In a forthcoming note in the Proceedings, the authors point out that their results for $k=3$ give a single formulation for the four classes into which Protter's basic set was divided.

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