A THREE PARAMETER FAMILY OF PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS WITH TWO ASSOCIATE CLASSES

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1. Introduction. A null polarity in $PG(2h+1, \rho^n) \equiv S$ will be used to show the existence of a design $D \equiv D(s, h, \rho^n)$ $(s = 1, 2, \cdots, h)$ with parameters

\begin{align*}
v &= q_{2h+1}, \\
b &= q_{2h+1}q_{2h-1}\cdots q_{2h-2s+1}/q_{2h-1}q_{2h-3}\cdots q_0, \\
r &= q_{2h-s}b/q_{2h+1}, \\
k &= q_{2h-1}, \\
n_1 &= q_{2h} - 1, \\
\lambda_1 &= q_{2h-s}q_{2h-s-1}b/q_{2h+1}q_{2h-1}, \\
n_2 &= q_{2h+1} - q_{2h}, \\
\lambda_2 &= \lambda_0 = 0,
\end{align*}

where $q_i = 1 + \rho^i + \cdots + \rho^{in}$. The special cases $D(1, 1, 2)$ and $D(1, 1, \rho^n)$ were derived by Bose and Shimamoto [3] and Clatworthy [4]. A method of constructing the design is indicated.

2. Definitions. A partially balanced incomplete block design with two associate classes having the parameters $v, b, r, k, \lambda_i, n_i, p_{jk}^i$ $(i, j, k = 1, 2)$ is an arrangement of $v$ varieties into $b$ blocks, together with a relationship called association defined for every pair of varieties satisfying the following conditions:

(i) Each of the $v$ varieties occurs in $r$ blocks. Each block contains $k$ varieties and no variety appears more than once in a block.

(ii) Every pair of varieties are either first or second associates. Each variety has $n_1$ first associates and $n_2$ second associates.

(iii) Any pair of varieties which are $i$th associates occur together in exactly $\lambda_i$ blocks $(i = 1, 2)$.

(iv) For every pair of varieties which are $i$th associates, the number of varieties common to the $j$th associates of the first and the $k$th associates of the second in $p_{jk}^i$ $(i, j, k = 1, 2)$.

It has been shown by Bose and Nair [2] that the parameters $p_{11}, n_i (i = 1, 2)$ determine the matrices $P_i (i = 1, 2)$ uniquely. The following relations are proved:

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(2.1) \( v_r = bk, \)
(2.2) \( v = n_1 + n_2 + 1, \)
(2.3) \( \lambda_1 n_1 + \lambda_2 n_2 = r(k - 1), \)
(2.4) \( p_{12}^1 = p_{21}^1 = n_1 - p_{11}^1 - 1, \)
(2.5) \( p_{22}^1 = n_2 - n_1 + p_{11}^1 + 1, \)
(2.6) \( p_{12}^2 = p_{21}^2 = n_1 - p_{11}^2, \)
(2.7) \( p_{22}^2 = n_2 - n_1 + p_{11}^2 - 1. \)

A nonsingular null polarity \( \Pi \) defined in \( S \) is a 1-1 incidence preserving correspondence between the points and \( 2h \)-spaces of \( S \) such that if \( x \) corresponds to the \( 2h \)-space \( \Pi(x) \), then \( x \in \Pi(x) \). Let \( x_i \) \((i = 0, 1, \ldots, s)\) be linearly independent points of an \( s \)-space \( \Sigma = \Sigma(x_0, x_1, \ldots, x_s) \subset S \). Then under the correspondence \( \Pi \) the \( s \)-space \( \Sigma \) corresponds to the \((2h-s)\)-space \( \Pi(x_0, x_1, \ldots, x_s) = \Pi(x_0) \cap \Pi(x_1) \cap \cdots \cap \Pi(x_s) \). Two points \( x, y \in S \) are called conjugate with respect to \( \Pi \) if \( x \in \Pi(y) \). It follows that \( y \in \Pi(x) \) and the relation is symmetric. In particular every point of \( S \) is self-conjugate with respect to \( \Pi \). It is possible to find \( s+1 \) linearly independent elements \( x_i \in S \) \((i = 0, 1, \ldots, s)\), \( s \leq h \), which are conjugate in pairs. In fact we can choose \( x_1 \in \Pi(x_0), x_2 \in \Pi(x_0, x_1), \ldots, x_s \in \Pi(x_0, x_1, \ldots, x_{s-1}) \). In the future the notation \( \Pi(x_0, x_1, \ldots, x_s) \) will indicate that the points \( x_i \) \((i = 0, 1, \ldots, s)\) are linearly independent and conjugate in pairs with respect to \( \Pi \).

3. Existence of \( D(s, h, p^n) \).

Theorem. Let the points of \( S \) be called varieties and let the \((2h-s)\)-spaces \( \Pi(x_0, x_1, \ldots, x_s) \) be called blocks. Let \( x, y \) be called first associates if they are conjugate with respect to \( \Pi \) and second associates otherwise. Then the varieties and blocks with the given definition of association may be abstractly identified with a design \( D(s, h, p^n) \) having the parameters stated in §1.

We first observe that varieties and pairs of varieties of the same type are symmetric in \( S \) so that incidences are independent of the particular element or pair of elements of the same type.

The number of varieties of \( D \) is equal to the number of points of \( S \), i.e. \( v = q^{2h+1} \).

The number of blocks is the number of \((2h-s)\)-spaces \( \Pi(x_0, x_1, \ldots, x_s) \). The number of ways of choosing \( s+1 \) elements conjugate in pairs is

\[
M = q^{2h+1}(q^{2h} - 1)(q^{2h-1} - q_1) \cdots (q^{2h-s+1} - q_{s-1})/(s + 1)!. 
\]
Let $\Pi (B)$ be the $s$-space corresponding to any block $B$. Then $B$ is the intersection of all $2h$-spaces $\Pi (y)$ with $y \in \Pi (B)$. In particular if $y_i \ (i=0,1,\ldots,s)$ are linearly independent points of $\Pi (B)$ then $\Pi (y_0) \cap \Pi (y_1) \cap \cdots \cap \Pi (y_s) = B$. The points $y_i \ (i=0,1,\ldots,s)$ are conjugate in pairs since $\Pi (B) \subseteq B$ for $s \leq h$. Thus $B$ may be represented in the form $\Pi (y_0, y_1, \ldots, y_s)$ where $y_i \ (i=0,1,\ldots,s)$ are any linearly independent points of $\Pi (B)$. It follows that in the enumeration $\Pi (x_0, x_1, \ldots, x_s)$ any $(2h-s)$-space is enumerated $N$ times where

$$N = \frac{q_s(q_s - 1)(q_s - q_1) \cdots (q_s - q_{s-1})}{(s + 1)!}.$$ 

Hence $b = M/N$.

The number of varieties in each block is $k = q^{2h-s}$ since this is the number of points in a $(2h-s)$-space. No variety appears more than once in a block.

The number of varieties conjugate to $x$ is the number of points of $\Pi (x)$ different from $x$, i.e. $n_1 = q_{2h-1} - 1$.

If $x, y$ are second associates, $y \in \Pi (x)$ so that $y \in \Pi (x) \cap \Pi (y)$. This implies $\lambda_1 = 0$.

Let $x, y$ be points of $S$. The points common to the first associates of $x$ and the first associates of $y$ must be points of $E = \Pi (x) \cap \Pi (y)$. If $x, y$ are first associates they both belong to $E$ so that $p_{11} = q_{2h-1} - 2$. If $x, y$ are second associates $x, y$ do not belong to $E$ so that $p_{11} = q_{2h-1}$.

The remaining parameters follow directly from equations 2.1-2.7.

4. **Construction of the design.** All nonsingular null polarities are projectively equivalent so that the designs obtained from different polarities are equivalent. It is sufficient then to use any null polarity in the construction.

Let $A$ be any nonsingular skew-symmetric $(2h+2) \times (2h+2)$ matrix with coefficients in $GF(p^n)$. In some allowable coordinate system any point $x$ of $PG(2h+1,p^n)$ may be represented by a $(2h+2) \times 1$ matrix $X \neq 0$. $\rho X (\rho \in GF(p^n), \rho \neq 0)$ represents the same point or variety. The $2h$-space corresponding to $X$ under the polarity $\Pi$ defined by $A$ is $U = AX$. $\rho U = A \rho X$ represents the same $2h$-space. The points $y$ of $U$ satisfy the equation $Y'AX = 0$ or $X'AY = 0$. Two points $x, y$ represented by the matrices $X, Y$ are thus conjugate whenever $X'AY = 0$. In particular $X'AX = 0$ since $A$ is skew symmetric.

The sets $\Pi (x_0, x_1, \cdots, x_s)$ may be found inductively. The sets $\Pi (x_0)$ are the sets of points $y$ satisfying the equation $Y'AX_0 = 0$ where $x_0$ is any point of $S$. Suppose the points $\Pi (x_0, x_1, \cdots, x_{s-1})$ have been constructed. Let $x_i$ be any point of $\Pi (x_0, x_1, \cdots, x_{s-1})$ not lying
in the \((s-1)\)-space \(\Sigma(x_0, x_1, \ldots, x_{s-1})\). Then \(\Pi(x_0, x_1, \ldots, x_s) = \Pi(x_0, x_1, \ldots, x_{s-1}) \cap \Pi(x_s)\).

It is sometimes convenient to represent the points by residue classes of integers. A convenient method of doing this is discussed in [1].

**Bibliography**


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