A NOTE ON INVARIANT SUBRINGS

GERTRUDE EHRLICH

The problem of invariant subrings has been studied in detail for certain types of rings satisfying the descending chain condition on one-sided ideals.

We prove here two theorems concerning invariant subrings of a ring $R$ without assuming the descending chain condition.

**Theorem 1.** Let $R$ be a ring with identity 1. If $S$ is a subring of $R$ with identity 1, and $S$ has a representation as the complete matrix ring of order $n^2 \geq 2$ over a ring with identity, then $S$ cannot be a proper invariant subring of $R$.

**Proof.** Let $E = \{e_{ij}\}$ be a set of $n^2$ matrix units contained in $S$, and let $B$ be the centralizer of $E$ relative to $S$. Then $S = B_n$. Let $A$ be the centralizer of $E$ relative to $R$. Then $R = A_n$, and $B \leq A$. If $t$ is an arbitrary element of $A$, we obtain (following Hattori [1]) that $e_{ii}(1+e_{ij}t)e_{ji}(1-e_{ji}t)e_{ii} = te_{ii}$ belongs to $S$ for arbitrary $i \neq j$. Hence $t = \sum_{i=1}^{n^2} te_{ii}$ belongs to $S$, and $A \subseteq S$. But then $A = B$, and $S = R$.

**Theorem 2.** Let $R$ be a ring with identity 1, and not of characteristic 2. Assume that $R$ has a representation as the complete matrix ring of order $n^2 \geq 2$ over a ring with identity. Let $S$ be an invariant subfield of $R$, with identity 1. Then $S$ is a subfield of the center of $R$.

**Proof.** Let $E = \{e_{ij}\}$ be a set of $n^2$ matrix units contained in $R$, and let $A$ be the centralizer of $E$ relative to $R$, so that $R = A_n$. We note first that for every noncentral element $x$ contained in $R$, there exists a square-nilpotent element $p \neq 0$ ($p^2 = 0$) such that $xp \neq px$. (For, if $x$ commutes with every square-nilpotent element, then $x$ commutes in $R$.)

Received by the editors, August 9, 1954.
particular with the elements $e_{ij}$ and $e_{ij}ye_{ij}$ for $i \neq j$, and $y$ arbitrary in $R$. This implies that $x$ belongs to the center of $R$.) Now, suppose $S$ is an invariant subfield of $R$, and $1 \in S$. If $S$ contains a noncentral element $x$, then $R$ contains a square-nilpotent element $p \neq 0$ such that $xp \neq px$. Let $t = 2 + p$. Then $t^{-1}$ and $(t - 1)^{-1}$ belong to $R$. Using Hua’s identity [2]

$$t = [x^{-1} - (t - 1)^{-1}x^{-1}(t - 1)] [t^{-1}x^{-1}t - (t - 1)^{-1}x^{-1}(t - 1)]^{-1}$$

and the hypotheses on $S$, we conclude that $t$ belongs to $S$. But then $p$ belongs to $S$. This is impossible since $p$ can have no inverse relative to the identity $1$.

References


University of Maryland