NONCOMMUTATIVE JORDAN ALGEBRAS OF CHARACTERISTIC 0

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Jordan algebras are commutative algebras satisfying the identity

\[(x^2a)x = x^2(ax).\]

These algebras have been studied extensively.

A natural generalization to noncommutative algebras is the class of algebras A satisfying (1). Linearization of (1), if the base field contains at least 3 elements, yields

\[(xy + yx, a, z) + (yz + zy, a, x) + (zx + xz, a, y) = 0\]

where \((x, y, z)\) denotes the associator \((x, y, z) = (xy)z - x(yz)\). If A contains a unity element 1, and if the characteristic is \(\neq 2\), then \(z = 1\) in (2) implies

\[(y, a, x) + (x, a, y) = 0,\]

or, equivalently,

\[(xa)x = x(ax).\]

That is, A is flexible (a weaker condition than commutativity). If a unity element is adjoined to A in the usual fashion, then a necessary and sufficient condition that (2) be satisfied in the extended algebra is that both (2) and (3) be satisfied in A.

We define a noncommutative Jordan algebra A over an arbitrary field \(F\) to be an algebra satisfying (1) and (4). These algebras include the best-known nonassociative algebras (Jordan, alternative, quasi-associative, and—trivially—Lie algebras). In 1948 they were studied briefly by A. A. Albert in [1, pp. 574–575], but the assumptions (1) and (4) seemed to him inadequate to yield a satisfactory theory, and he restricted his attention to a less general class of algebras which he called "standard." In this paper, using Albert's method of trace-admissibility and his results for trace-admissible algebras, we give a

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2 Numbers in brackets refer to the references cited at the end of the paper.

3 We use the formulation given in [2] as being more convenient for characteristic 0 than the modified version presented in [4]. We are limited to the characteristic 0

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structure theory for noncommutative Jordan algebras of characteristic 0.

1. Some elementary properties. It is shown in [1] that flexibility implies that (1) is equivalent to any one of the following:

\[ x^2(xa) = x(x^2a), \quad (xa)x^2 = x(ax^2), \quad (ax)x^2 = (ax^2)x, \]

and that (1) and (5) imply that \( A \) of characteristic \( \neq 2 \) is \textit{Jordan-admissible}; that is, the commutative algebra \( A^+ \) in which multiplication is defined by \( x \cdot y = (xy + yx)/2 \) is a Jordan algebra.

It seems not to have been noted that every flexible Jordan-admissible algebra of characteristic \( \neq 2 \) satisfies (1), so that \( A \) is a noncommutative Jordan algebra if and only if \( A \) is flexible and Jordan-admissible. For, denoting right and left multiplications in \( A \) by \( Rx \) and \( Lx \) respectively, \( (x^2 \cdot a) \cdot x = x^2 \cdot (a \cdot x) \) implies

\[ [R_x + L_x, Rx + L_x] = 0. \]

But (4) implies \( [L_x, R_x] = 0 \), while (3) implies both \( [L_y, R_x] = [R_y, L_x] \) and

\[ R_{xy} - R_x R_y = L_{yx} - L_x L_y. \]

In particular, we have \( [L_x, R_x] = [R_x, L_x] \), \( R_x - R_x^2 = L_x - L_x^2 \). Then

(6) gives \( 0 = [R_x + L_x, R_x] + [R_x + L_x, L_x] = [R_x^2 - L_x^2 + L_x^2, R_x] + [R_x^2 - L_x^2 + L_x^2, L_x] = 2[R_x, R_x] + 2[R_x, L_x] = 4[R_x, R_x], \) implying (1). It is not possible to derive (4) from Jordan-admissibility even for algebras containing a unity element, for there are known examples of Jordan-admissible algebras which are not flexible, but which do contain 1 [3, p. 186].

Any noncommutative Jordan algebra of characteristic \( \neq 2 \) is power-associative.\(^4\) For, defining \( x^{k+1} = x^k x \), we may prove \( x^k x^\mu = x^{k+\mu} \) by induction on \( \lambda + \mu = n \). Since this is true for \( n = 2, 3 \) by (4), we assume \( x^\lambda x^\mu = x^{\lambda + \mu} \) for all \( \lambda + \mu < n, \ n \geq 4 \). Let \( a = x^{n-3} \) in (5): \( x x^{n-1} = x^{n-2} x^2 = x^{n-1} x = x^n \). We need to prove \( x^n - ax^n = x^n \) for \( \alpha = 1, \cdots, n - 1, \) and prove this by proving \( x^n - ax^n = x^n = x^{\alpha} x^{n-\alpha} \) by induction on \( \alpha \). This has been proved for \( \alpha = 1 \), and we assume \( x^n - \beta x^n = x^n = x^\beta x^n - \beta \) for all \( \beta \leq \alpha < n - 1 \), and prove \( x^n - (\alpha + 1) x^{\alpha + 1} = x^n = x^{\alpha + 1} x^{n-\alpha} \) as follows. Let

\(^{4} \) This is proved for characteristic \( \neq 2, 3, 5 \) in [1, p. 574].
\[ a = x^{n-(a+2)}, \quad y = x, \quad z = x^a \] in (2). Since the sum of any three or fewer exponents is \( < n \), we have
\[ 0 = 2(x^2, a, x^a) + 4(x^{a+1}, a, x) = 2x^{n-a}x^a - 2x^2x^{n-2} + 4x^{n-1}x - 4x^{a+1}x^{n-(a+1)} = 4x^n - 4x^{a+1}x^{n-(a+1)}. \]
But (3) implies \( 0 = 2(x^a, a, x^2) + 4(x, a, x^{a+1}) = 4x^n - 4x^{n-(a+1)}x^{a+1}. \)

2. **Trace-admissibility.** A bilinear function \( \tau(x, y) \) on a power-associative algebra \( A \) and with values in the base field \( F \) is called an **admissible trace function** for \( A \) (and \( A \) is called **trace-admissible**) in case
- (i) \( \tau(x, y) = \tau(y, x) \),
- (ii) \( \tau(xy, z) = \tau(x, yz) \),
- (iii) \( \tau(e, e) \neq 0 \) for any idempotent \( e \) of \( A \),
- (iv) \( \tau(x, y) = 0 \) if \( xy \) is nilpotent.

If \( A \) contains 1, then the use of a linear function \( \delta(x) = \tau(1, x) \) may be substituted for that of \( \tau(x, y) \), and it is known [2, p. 319] that (i)-(iv) are equivalent to:
- (I) \( \delta(xy) = \delta(yx) \),
- (II) \( \delta((xy)z) = \delta(x(yz)) \),
- (III) \( \delta(e) \neq 0 \) for any idempotent \( e \) of \( A \),
- (IV) \( \delta(x) = 0 \) if \( x \) is nilpotent.

**Theorem.** Any noncommutative Jordan algebra \( A \) of characteristic 0 is trace-admissible.

**Proof.** Since any subalgebra of a trace-admissible algebra is trace-admissible, it is sufficient to prove this theorem under the assumption that \( A \) has a unity element 1. It is well known that, since the characteristic is 0,

\[ \delta(x) = \text{Trace } R_x^+ = (1/2) \text{ Trace } (R_x + L_x) \]

is an admissible trace function for the Jordan algebra \( A^+ \). We shall show that \( \delta(x) \) in (8) is also an admissible trace function for \( A \).

Since powers in \( A \) and \( A^+ \) coincide, (III) and (IV) are valid in \( A \) because they hold in \( A^+ \). Flexibility implies (I), for \( \delta(xy) - \delta(yx) = (1/2) \text{ Trace } (R_{xy} + L_{xy} - R_yz - L_{yZ}) = (1/2) \text{ Trace } (R_xR_y - R_yR_x + L_yL_z - L_zL_y) = 0 \) by (7). Now (II) in \( A^+ \) implies

\[ \delta((z \cdot y) \cdot x) = \delta(z \cdot (y \cdot x)). \]

Hence (3), (I), and (9) yield
\[ 4\delta((xy)z) = \delta(2(xy)z + 2(xy)z) = \delta(2x(yz) - 2(yz)x + 2z(xy) + 2(xy)z) = \delta(4x(yz) - x(yz) - (yz)x - x(yz) + z(xy) + (yx)z + (xy)x + z(xy)) = 4\delta(x(yz)) - 4\delta((z \cdot y) \cdot x) + 4\delta(z \cdot (y \cdot x)) = 4\delta(x(yz)), \]

implying (II).

3. **Structure theory.** Because of the general results in [2], the structure theory for noncommutative Jordan algebras of characteristic 0
may be given as a set of corollaries to our theorem in §2. Define the radical $N$ of $A$ to be the maximal nilideal\(^6\) of $A$, and $A$ to be semisimple if its radical is 0. Define $A$ to be simple if $A$ is simple in the ordinary sense and not a nilalgebra.

**Corollary 1.** The radical $N$ of any noncommutative Jordan algebra $A$ of characteristic 0 is the set of all $z$ satisfying $\delta(xz) = 0$ for every $x$ in $A$, and coincides with the radical of the Jordan algebra $A^+$. Also $A/N$ is semisimple.

**Corollary 2.** Any semisimple noncommutative Jordan algebra $A$ of characteristic 0 is uniquely expressible as a direct sum $A = A_1 \oplus \cdots \oplus A_t$ of simple ideals $A_i$.

**Corollary 3.** The simple noncommutative Jordan algebras of characteristic 0 are

(a) the simple (commutative) Jordan algebras,
(b) the simple flexible algebras of degree two,
(c) the simple quasiassociative algebras.

The simple quasiassociative algebras are determined in [1, Chap. V] and are further studied in [5]. A set of necessary and sufficient conditions for the multiplication table of any simple flexible algebra of degree two is given in [1, p. 588] but, inasmuch as these algebras include such interesting examples as the Cayley-Dickson algebras and the $2^t$-dimensional algebras obtained by the Cayley-Dickson process [6], a complete determination of those which are not commutative remains an interesting problem.

**References**


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\(^6\) One cannot hope to prove that this radical is nilpotent, or even solvable. For every Lie algebra is its own radical by this definition.