A NOTE ON TAUBERIAN CONDITIONS FOR ABEL AND CESÁRO SUMMABILITY

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A conjecture has recently been made by P. L. Butzer\(^1\) that there exist Tauberian conditions for the Cesàro method of summability which are not also Tauberian conditions for Abel summability.

It was pointed out\(^2\) by G. Lorentz that the conjecture is true in a trivial sense if we define the Tauberian condition as membership of a class \(T\) consisting of all series satisfying one of the classical Tauberian conditions for Cesàro summation \((na_n \geq -K\), for example\) together with one particular series which is Abel, but not Cesàro, summable. It does not seem possible to restrict the class of admissible Tauberian conditions in a natural way so that examples of this rather artificial kind are excluded.

However, even if Butzer’s question does not admit a nontrivial yes or no answer, it still has substance if we look for a solution with some intrinsic formal interest.

The condition \((\tau)\) defined below applies to the slightly more general summation methods defined by integral transforms of functions of a continuous real variable. These reduce to the classical methods of summation of series when the functions are step functions with jumps at integral values. Thus, if \(s(x)\) is defined for \(x \geq 0\) and is integrable over any finite interval, it tends to \(A\) in the Abel sense if

\[
\lim_{\delta \to 0} \delta \int_0^\infty e^{-\delta x} s(x) \, dx = A,
\]

it being assumed that the integral converges absolutely when \(\delta > 0\).

It tends to \(A\) in the Cesàro sense if

\[
\lim_{X \to \infty} \frac{1}{X} \int_0^X s(x) \, dx = A.
\]

A Tauberian condition for Abel or Cesàro summability is one which, together with (1) or (2), respectively, ensures that \(s(x) \to A\).

We shall say that \(s(x)\) satisfies the condition \((\tau)\) if, for every

\(^1\) Bull. Amer. Math. Soc. Research Problem 60-3-14.
\(^2\) In a letter to the editors received July 14, 1954.
\( \epsilon > 0 \), there is a positive number \( \eta(\epsilon) \), independent of \( x \), such that, for all sufficiently large \( x \),

\[
\left| \int_x^{RX} [s(y) - s(x)] dy \right| \leq (R - 1)\epsilon X
\]

for some \( X(\epsilon, x), R = R(\epsilon, x) \) satisfying

\[
R \geq 1 + \eta, \\
xR^{-1} \leq X \leq x.
\]

The essential point about condition (\( \tau \)) is that it restricts the mean values of \( s(y) - s(x) \) and not \( |s(y) - s(x)| \), as do the familiar classical conditions. It is easy to show that the stronger form of (\( \tau \)) with the modulus inside the integral is a Tauberian condition for Abel summation and therefore not an answer to Butzer's problem.

In the case of a step function \( s(x) = \sum_{n \leq x} a_n \), it is easy to show that (\( \tau \)) is implied by the classical Tauberian condition \( na_n \geq -K \) for some positive constant \( K \) or by the "high indices" condition.

**Theorem 1.** Condition (\( \tau \)) is a Tauberian condition for Cesàro summability.

Writing

\[
C(X) = \frac{1}{X} \int_0^X s(x) dx,
\]

it follows from (3) that

\[
\left| s(x) - \frac{R}{R - 1} C(RX) + \frac{C(X)}{R - 1} \right| < \epsilon
\]

for large \( x \) and assuming, by a trivial transformation, that \( A = 0 \), \( C(x) \to 0 \), it follows that

\[
\limsup |s(x)| < \epsilon, \quad s(x) \to 0.
\]

To conclude, we show that (\( \tau \)) is not an Abel-Tauberian condition by proving:

**Theorem 2.** There exists a function \( s(x) \) satisfying (\( \tau \)) which does not tend to a limit as \( x \to \infty \) but which tends to a limit in the Abel sense.

We write

\[
\lambda_m = (2m + 1) \log(2m + 1) \quad (m = 0, 1, 2, \ldots)
\]

and define
It is plain that $s(x)$ does not tend to a limit, but
\[
\delta \int_0^\infty e^{-\delta x} s(x) \, dx = \int_0^\infty e^{-\delta x} s(x) \, dx = -3 \sum_{m=0}^\infty e^{-\delta (2m+1)} \log (2m+1)(-2)^{m-1}
\]
\[
= -3 \sum_{n=1}^\infty e^{-\delta n} \log n2^{(n-3)/2} \cos (n + 1)\pi/2
\]
\[
= R \left[-3, 2^{-3/2} e^{\pi i/2} \sum_{n=1}^\infty e^{-\delta n} \log (21/2 e^{\pi i/2})^n\right],
\]
and it is known\(^8\) that this tends to a limit as $\delta \to +0$.

Finally, we have to show that $s(x)$ satisfies (\tau). For every positive $\epsilon$, we define
\[
X(\epsilon, x) = \lambda_M,
\]
where $M = M(x)$ is determined by
\[
\lambda_M \leq x < \lambda_{M+1}.
\]
We shall suppose that $M$ is even. The odd case needs only trivial changes. Next we define the even integer $P = P(x)$ by
\[
\lambda_{P-1} \leq 2\lambda_M < \lambda_{P+1}.
\]
For all positive $\epsilon$, we shall define $R(\epsilon, x) = R(x)$ so that
\[
\lambda_P \leq R\lambda_M < \lambda_{P+1}.
\]
Since $\lambda_{m+1} - \lambda_m = o(\lambda_m)$, it follows from (10) and (11) that $R(x) \to 2$ as $x \to \infty$; and therefore (4) holds. Now
\[
\int_{\lambda_M^P}^{\lambda_P} s(y) \, dy = \sum_{m=M}^{P-1} (\lambda_{m+1} - \lambda_m)(-2)^m
\]
\[
< (-2)^{P-1}(\lambda_P - \lambda_{P-1}) \left(1 - \sum_{v=1}^{P-M-1} 2^{-v}\right)
\]
since $P$ is even and $\lambda_{m+1} - \lambda_m$ increases with $m$. This expression is negative, and therefore
\[
\int_{\lambda_M^P}^{\lambda_P} s(y) \, dy < 0 < s(x).
\]

\(^8\) Lindelöf, Journal de Mathématiques (5) vol. 9 (1903) pp. 213–221. See also Hardy, Divergent series, Oxford, 1949, Theorem 32, p. 78.
A similar argument shows that
\[ \int_{\lambda_M}^{\lambda_{P-1}} s(y) \, dy > 0, \]
so that
\[ \int_{\lambda_M}^{\lambda_{P-1}} s(y) \, dy > \int_{\lambda_{P-1}}^{\lambda_{P-1}} s(y) \, dy = 2^P(\lambda_{P+1} - \lambda_P) - 2^{P-1}(\lambda_P - \lambda_{P-1}) \]
\[ > 2^{P-1}(\lambda_{P+1} - \lambda_P) > \frac{2^{P-M-1}}{P - M + 1}(\lambda_{P+1} - \lambda_M)s(x) \]
since \(\lambda_{m+1} - \lambda_m\) increases and \(s(x) = 2^M\). Moreover, since \(\lambda_{m+1} - \lambda_m = o(\lambda_m)\), it follows from (10) that \(P - M \to \infty\) as \(x \to \infty\) and so \(2^{P-M-1} > P - M + 1\) for large \(x\). Hence, writing \(X\) for \(\lambda_M\),
\[ \frac{1}{(R - 1)X} \int_X^{RX} s(y) \, dy > s(x) \]
when \(XR = \lambda_{P+1}\). Since the expression on the left of (13) is continuous in \(R\) for \(\lambda_P \leq XR \leq \lambda_{P+1}\), and is less than \(s(x)\) when \(XR = \lambda_P\), by (12), it follows that
\[ \frac{1}{(R - 1)X} \int_X^{RX} s(y) \, dy = s(x) \]
for some \(R\) in the range defined by (11). If we define \(R(x)\) to be this value, the condition \((\tau)\) is satisfied.

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