PERMUTATIONS PRESERVING CONVERGENCE OF SERIES

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1. Introduction. We shall solve a problem involving rearrangements of series which was proposed to the author by R. Creighton Buck. The problem is to characterize those permutations $p_1, p_2, p_3, \ldots$ of the integers $1, 2, 3, \ldots$ such that the two series in

$$\sum_{k=1}^{\infty} a_{p_k} = \sum_{k=1}^{\infty} a_k$$

converge to equal values whenever $\sum a_k$ is a convergent series of real (or complex) terms. We shall show that a given permutation $p_1, p_2, \ldots$ has the required property if and only if there is an integer $N$ such that for each $n = 1, 2, 3, \ldots$ the set of integers which appear among the integers $p_1, p_2, \ldots, p_n$ is representable as the union of $N$ or fewer blocks of consecutive integers.

2. Proof. Let $p_1, p_2, \ldots$ be a given permutation of the integers $1, 2, 3, \ldots$. Let

$$s(n) = \sum_{k=1}^{n} a_k, \quad \sigma(n) = \sum_{k=1}^{n} a_{p_k}.$$

Suppose that $n$ is so great that the integer 1 is included in the set $p_1, p_2, \ldots, p_n$. Then the integers in the set $p_1, p_2, \ldots, p_n$ are, in increasing order,

$$1, *, 1, \ldots, po, ai, 1, ai + 1, ai + 1, \ldots, pi - 1, pi - 1, \ldots, \alpha_0, \alpha_1, \ldots, \beta_0, \alpha_2, \beta_1, \alpha_3, \beta_2, \alpha_4, \beta_3, \ldots, \beta_{n-1}, \alpha_n, \beta_n$$

where

$$0 < \beta_0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \beta_{n-1} < \alpha_n.$$

Hence

$$\sigma(n) = s(\beta_0) + [s(\beta_1) - s(\alpha_1)] + \cdots + [s(\beta_{n-1}) - s(\alpha_{n-1})]$$

and therefore

$$\sigma(n) = s(\beta_0) - s(\alpha_1) + s(\beta_1) - s(\alpha_1) + \cdots + s(\beta_{n-1}) - s(\alpha_{n-1}) + s(\beta_{n-1}).$$

Since $\lim_{n \to \infty} \beta_0 = \infty$, this has the standard form

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\[ \sigma(n) = \sum_{k=1}^{\infty} a_{nk} s(k) \]

where\:
\[
\lim_{n \to \infty} a_{nk} = 0, \quad k = 1, 2, 3, \ldots ,
\]

and
\[
\sum_{k=1}^{\infty} a_{nk} = 1, \quad n = 1, 2, 3, \ldots .
\]

This transformation from \( s(n) \) to \( \sigma(n) \) is regular, that is, such that existence of \( \lim s(n) \) implies \( \lim \sigma(n) = \lim s_n \), if and only if there is a constant \( M \) for which
\[
\sum_{k=1}^{\infty} |a_{nk}| \leq M.
\]

We see that
\[
\sum_{k=1}^{\infty} |a_{nk}| = 2B_n - 1
\]

where \( B_n \) is the number of disjoint blocks of consecutive integers appearing in the set \( p_1, p_2, \ldots, p_n \) and hence that existence of \( \lim s(n) \) implies \( \lim \sigma(n) = \lim s(n) \), if and only if there is a constant \( M \) for which \( 2B_n - 1 \leq M \) or \( B_n \leq (M+1)/2 \). This proves the result.

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