NOTE ON THE FIRST CESÀRO MEAN OF THE DERIVED CONJUGATE SERIES OF FOURIER SERIES

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1. Let \( f(t) \) be integrable \( L \) in \((-\pi, \pi)\) and periodic with period \( 2\pi \) and let

\[
(1.1) \quad f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \frac{1}{2} a_0 + \sum_{1}^{\infty} A_n(t).
\]

The conjugate series of (1.1) at \( t=x \) is

\[
(1.2) \quad \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{1}^{\infty} B_n(x).
\]

Then the differentiated series of (1.2) is

\[
(1.3) \quad -\sum_{n=1}^{\infty} n(a_n \cos nx + b_n \sin nx) = -\sum_{1}^{\infty} nA_n(x) = -\sum_{1}^{\infty} L_n, \text{say.}
\]

We write \( \phi(t) = f(x+t) + f(x-t) - 2f(x), \) \( h(t) = \phi(t) / t - d, \) where \( d=d(x) \). Let \( t_n \) be the \( n \)th Cesàro mean of order 1 of the series (1.3). The object of the present note is to prove the following

**Theorem.** If

\[
(1.4) \quad \int_{t}^{\pi} \frac{|h(u)|}{u} \, du = o \left( \log \frac{1}{t} \right), \quad \text{as} \ t \to 0,
\]

then

\[
t_n / \log n \to d/\pi, \quad \text{as} \ n \to \infty.
\]

The relation between Condition (1.4) and the Condition

\[
(1.5) \quad \int_{0}^{t} |h(u)| \, du = o(t)
\]

is brought out by the following

**Lemma.** If \( \int_{0}^{t} |h(u)| \, du = o(t), \) as \( t \to 0, \) then

\[
\int_{t}^{\pi} \frac{|h(u)|}{u} \, du = o \left( \log \frac{1}{t} \right).
\]

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1 Throughout the present note it is assumed that \( h(t) \) is \( L \) in \((0, \pi)\).
On the other hand, if \( \int_t^t |h(u)|/u \, du = o(\log (1/t)) \) then

\[
\int_0^t |h(u)| \, du = o\left(t \log \frac{1}{t}\right).
\]

This is known.\(^2\)

If \( k_2(n, t) \) denotes the \( n \)th Cesàro mean of order 2 of the sequence \( 2\pi x \sin nt \), then we have the following inequalities:\(^3\)

\[
\begin{align*}
(1.6) & \quad \left| \frac{d}{dt} k_2(n, t) \right| \leq A n, \\
(1.7) & \quad \leq An^{-1} t^{-2}.
\end{align*}
\]

\( (1.6) \) is trivial, and \( (1.7) \) is known.\(^4\)

2. Proof of the Theorem. Now

\[
L_k = kA_k(x) = \frac{k}{\pi} \int_0^x \phi(t) \cos ktdt
\]

\[
(2.1) = \frac{k}{\pi} \int_0^x \frac{\phi(t)}{t} t \cos ktdt = \frac{k}{\pi} \int_0^x \{h(t) + d\} t \cos ktdt
\]

\[
= \frac{1}{\pi} \int_0^x th(t)k \cos ktdt + \frac{d}{\pi} \int_0^x tk \cos ktdt
\]

\[
= \beta_k + \gamma_k, \quad \text{say.}
\]

Integrating by parts, we get

\[
(2.2) \quad \gamma_k = -\frac{d}{\pi} \frac{1 - (-1)^k}{k} = -\frac{d}{\pi} \omega_k, \quad \text{say.}
\]

Hence

\[
-\tau_n = \frac{1}{n+1} \sum_{k=1}^n (n - k + 1)L_k = \frac{1}{n+1} \sum_{n+1}^n (n - k + 1)\beta_k
\]

\[
+ \frac{1}{n+1} \sum_{1}^n (n - k + 1)\gamma_k = \tau_n^{(1)} + \tau_n^{(2)}, \quad \text{say.}
\]

Now

\(^2\) M. L. Misra, Quart. J. Math. Oxford Ser. vol. 18 (1947) p. 149 with \( \phi \) replaced by \( h \).

\(^3\) A is used to denote a number independent of \( n \) and \( t \); but its value may differ from occurrence to occurrence.

\[
\begin{align*}
    t_n^{(1)} &= \frac{1}{n+1} \frac{1}{\pi} \int_0^\pi \text{th}(t) \left( \frac{d}{dt} \right) \sum_{k=1}^n (n - k + 1) \sin kt \, dt \\
    &= \frac{n+2}{4} \int_0^\pi \text{th}(t) \left( \frac{d}{dt} \right) k_3(n, t) \, dt \\
    &= \frac{n+2}{4} \left\{ \int_0^{\pi/n} + \int_{\pi/n}^\pi \right\} \\
    &= l_1 + l_2, \text{ say.}
\end{align*}
\]

Using the lemma, (1.6), and (1.7), it is easy to see that

\[
    l_1 = o(\log n), \\
    l_2 = o(\log n),
\]

so that

\[
(2.5) \quad t_n^{(1)} = o(\log n).
\]

Lastly we consider \( t_n^{(2)} \). By (2.2) and (2.3), we have

\[
(2.6) \quad t_n^{(2)} = -\frac{d}{\pi} \frac{1}{n+1} \sum_{k=1}^n (n - k + 1) \omega_k.
\]

By partial summation we find\(^6\)

\[
\begin{align*}
    \frac{1}{n+1} \sum_{k=1}^n (n - k + 1) \omega_k \\
    &= \frac{1}{n+1} \sum_{k=1}^n \Omega_k \\
    &= \frac{1}{n+1} \sum_{k=1}^n (\log k + C + E_k) \\
    \quad = \frac{1}{n+1} \log \Gamma(n+1) + C + o(1) \\
    \quad = \frac{1}{n+1} \left\{ \left( n + \frac{1}{2} \right) \log (n+1) - (n+1) + C_1 + o(1) \right\} \\
    \quad + C + o(1) \quad \sim \log n.
\end{align*}
\]

\(^6\) Here \( \Omega_k = \sum_{r=1}^k \omega_r = \log k + C + E_k \), where \( C \) is a constant and \( E_k \to 0 \) as \( k \to \infty \).

Thus by (2.6) and (2.7)

\[(2.8) \quad t_n^{(2)} \sim -\frac{(d/\pi)}{\log n} \]

Hence by (2.3), (2.5), and (2.8), we have

\[t_n \sim \frac{(d/\pi)}{\log n},\]

which completes the proof.\(^7\)

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\(^7\) A note dealing with the analogous result for the differentiated Fourier series proved under the condition corresponding to (1.5) has been accepted by this journal. This condition may be replaced by the one corresponding to (1.4).