THE POSITION OF C-SETS IN SEMIGROUPS

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A continuum is a compact connected Hausdorff space. If X is a continuum, then C ⊂ X is a C-set if (i) C ≠ X, (ii) for each subcontinuum A ⊂ X that meets C we have either A ⊂ C or C ⊂ A. Thus a composant of a solenoid is a C-set.

A clan is a Hausdorff space together with a continuous associative multiplication. A clan is a compact connected mob with (two-sided) unit, denoted by u. In what follows we assume that S is a mob.

(*) Assume that S is compact. Let E be the set of idempotents of S and let K be the minimal ideal of S [4]. We know that each e ∈ E is contained in a maximal subgroup H(e) of S and that no two such maximal subgroups intersect [8, Theorem 1]. Moreover ([1] and [8, Theorem 3]),

\[ K = \bigcup \{ H(e) \mid e \in E \cap K \}, \]

and, if e ∈ E, then e ∈ K if and only if H(e) = eSe. Finally, \[ K = \bigcup \{ L \mid L \in \mathcal{L} \} \]
where \( \mathcal{L} \) is the set of minimal left ideals of S and \( L \in \mathcal{L} \) if and only if \( L = Sx \) for some \( x \in K \) [1].

We refer to the above paragraph as (*). Its content is essential for the proofs of most of the theorems below. In some cases we may cite (*) and use instead the left-right dual of an assertion in (*).

Our purpose in this note is to examine the position of C-sets in clans relative to K and to the sets H(e), e ∈ E. We show, for example, that if C is a C-set meeting H(e) ≠ \{e\}, then C ⊂ H(e). As noted earlier, topological groups may contain C-sets and it is easy to give examples of clans which are not groups and which contain C-sets. The results here are more decisive than those in [6].

The following lemma may be presumed to lie in the public domain, and we are unable to give any references.

**Lemma 1.** Let X be a continuum and let C be a C-set of X. Then C is connected and contains no inner points.

The proof of the lemma consists of repeated applications of the familiar result that, if X is a continuum and if U is an open subset with \( U^* \setminus U \neq \emptyset \), then each component C of U satisfies \( C^* \cap (U^* \setminus U) \neq \emptyset \).

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If \( X \) is a space and if \( \{ A_\lambda | \lambda \in \Lambda \} \) is a family of subsets of \( X \) indexed by a directed set \( \Lambda \), then we define two subsets of \( X \), \( \text{sup} \ A_\lambda \) and \( \text{inf} \ A_\lambda \), as follows: \( x \in \text{sup} \ A_\lambda \) if, for each open set \( U \) about \( x \), there exists a cofinal \( \{ \text{residual} \} \) subset \( \Lambda(U) \subset \Lambda \) such that \( U \cap A_\lambda \neq \emptyset \) for each \( \lambda \in \Lambda(U) \). We write \( \lim A_\lambda = A \), or \( A_\lambda \rightarrow A \) if \( \text{inf} \ A_\lambda = A = \text{sup} \ A_\lambda \). If \( \Lambda \) is the set of positive integers, then this notion of convergence is the usual one, given for example in detail in Kuratowski [3]. We shall suppose that the reader can supply the simple unstated results that we need concerning this notion.

**Lemma 2.** Let \( S \) be a compact mob, and let \( \{ A_\lambda | \lambda \in \Lambda \} \) and \( \{ B_\lambda | \lambda \in \Lambda \} \) be two families of subsets of \( S \) based on the same directed set \( \Lambda \). If \( A_\lambda \rightarrow A \) and \( B_\lambda \rightarrow B \), then \( A_\lambda \cdot B_\lambda \rightarrow A \cdot B \) [5].

As in [2, Lemma 2], we may prove

**Lemma 3.** If \( S \) is a continuum with left unit and if \( L \) is a left ideal of \( S \), then \( K \cup L \) is connected.

**Theorem 1.** Let \( S \) be a continuum with left unit, let \( L \) be a closed left ideal of \( S \), and let \( C \) be a \( C \)-set of \( S \). If \( C \) meets \( K \cup L \) then \( C \subset K \cup L \) (cf. [6, Theorem 1]).

**Proof.** By Lemma 3 we know that \( K \cup L \) is a continuum. We may assume that \( K \cup L \) is a proper subset of \( C \). Let \( p \in C \setminus (K \cup L) \) and let \( U \) be an open set about \( p \) such that \( U^* \) does not meet \( K \cup L \). Let \( M \) be the union of all left ideals of \( S \) contained in \( S \setminus U^* \). Then \( M \) is an open [2, Lemma 1] left ideal which is connected by Lemma 3. Now \( M^* \) is a continuum not containing \( p \) and intersecting \( C \). Hence \( M^* \subset C \), contrary to the fact that, by Lemma 1, \( C \) contains no inner points.

**Theorem 2.** If \( S \) is a clan and if \( K \) is a \( C \)-set, then \( K \) is a maximal subgroup of \( S \).

**Proof.** If \( S \) is a group then \( K = S \) and the result follows. If \( S \) is not a group then \( S \setminus H(u) \) is an ideal [2, Theorem 4] so that \( u \) is not in \( K \) and \( S \setminus K \) is not void. Recall that \( S \) is connected and that \( K \) has no inner points by Lemma 1. Let \( C \) be an idempotent in \( K \) (see (*) ) and let \( \{ a_\lambda | \lambda \in \Lambda \} \) be a directed set of points of \( S \setminus K \) with \( a_\lambda \rightarrow e \). Now \( a_\lambda \in a_\lambda S \) and \( a_\lambda S \) meets \( K \) because any right ideal meets any ideal. Since \( K \) is a \( C \)-set and \( a_\lambda \in S \setminus K \) we must have \( K \subset a_\lambda S \) for each \( \lambda \). Thus \( K \cap \{ a_\lambda S | \lambda \in \Lambda \} \subset \lim a_\lambda S = eS \), by Lemma 2. Thus \( K \supset eS \) and dually \( K \supset eSe \) so that \( K \subset eSe = eS \cap Se \). Now \( e \in K \) gives \( eSe \subset K \) and \( eSe = H(e) \) (see (*) ) and thus \( K = H(e) \).

This result may fail if \( S \) has only a left unit.
Theorem 3. If $S$ is a clan, if $e \in E$, and if $C$ is a $C$-set meeting $H(e) \neq \{e\}$, then $C \subseteq H(e)$.

Proof. Since $H(e) \subseteq eSe = eS \cap Se$ we know that $C$ intersects both $Se$ and $eS$.

(A) Let $C \cap K = \emptyset$. Now $eS$ is a continuum which intersects $K$ so that, of the two possibilities $C \subseteq eS$ and $eS \subseteq C$, we must have the former. Similarly $C \subseteq Se$ so that $C \subseteq eSe$. Now $eSe$ is a clan with unit $e$ and $H(e)$ is the maximal subgroup of $eSe$ containing $e$. By Theorem 2 of [6] and Lemma 1 it follows that $C \subseteq H(e)$.

(B) Let $C \cap K \neq \emptyset$ so that $C \subseteq K$ by Theorem 1. Thus $H(e)$ meets $K$ and so $H(e) \subseteq K$ since $H(e)$ is a subgroup of $S$. Since $H(e) \subseteq eS \cap Se$ both $eS$ and $Se$ intersect $C$. There are four possibilities which we state and consider.

(i) $eS \cup Se \subseteq C$. We know (see (*)) that each $L \in \mathcal{L}$ meets $eS$. For any such $L \neq Se$ we have $L \subseteq C$ since $L$ is a continuum ($L = Sx$ for some $x \in K$, see (*)) and $C \subseteq L$ would give $L \cap Se \neq \emptyset$, contrary to the fact that $L$ and $Se$ (see (*)) are minimal left ideals. Since $K$ is the union of the minimal left ideals of $S$ (see (*)) we know that $K \subseteq C$ and thus $C \subseteq K$ by Theorem 1. By Theorem 2, $K$ is a maximal subgroup of $S$, i.e., $K = H(e)$, so that $C \subseteq H(e)$.

(ii) $eS \subseteq C \subseteq Se$. With the notation of (*) each $L \in \mathcal{L}$ intersects $eS$ and thus $Se$ and hence $L = Se$. It follows by (*) that $K = Se$. By the dual of Theorem 1 of [7], and that part of the proof of Theorem 4 of [7] on the middle of page 53, we know that there is a topological isomorphism $\phi: K \to H(e) \times (E \cap K)$. If $E \cap K$ is degenerate then $K = H(e)$ and $C \subseteq H(e)$. If $E \cap K$ contains more than one element then ($H(e) \neq \{e\}$ by assumption) the cartesian product of two nondegenerate continua contains a $C$-set, $\phi(C)$. Now a somewhat dull argument shows that $\phi(C)$ is a point and thus $C$ is a point and hence $C \subseteq H(e)$.

(iii) $Se \subseteq C \subseteq eS$. This is the dual of (ii).

(iv) $C \subseteq Se$ and $C \subseteq eS$. Then $C \subseteq eS \cap Se = eSe = H(e)$ since $e \in K$ and using (*).

Although our last theorem does not involve $C$-sets, it fits easily into the pattern of this note.

With the aid of Theorem 4 and Lemma 3 of [2] as well as the argument used in the proof of these results we can prove

Lemma 4. If $S$ is a clan with unit $u$ and if $H(u)$ contains an inner point, then $H(u) = S$.

Theorem 4. If $S$ is a continuum, if $e \in K$, and if $H(e)$ has an inner point, then $H(e) = K$. 
Proof. It is clear that $eSe$ is a clan with unit $e$, that $H(e)$ is the maximal subgroup of $eSe$ containing its unit and that $H(e)$ has inner points relative to $eSe$. Thus $H(e) = eSe$ by Lemma 4 and by (*) $H(e) \subseteq K$. Suppose that $H(e) \neq K$. By Lemma 3, $K$ is connected so that there is a directed set of points $\{x_\lambda | \lambda \in \Lambda \}$ of $K \setminus H(e)$ with $x_\lambda \rightarrow x \in H(e)$. It follows from Lemma 2 that $x_\lambda Sx_\lambda \rightarrow xSx = eSe = H(e)$. Since $H(e)$ contains an inner point we know by definition that $H(e) \cap x_\lambda Sx_\lambda \neq \emptyset$ for some $\lambda \in \Lambda$. Now $x_\lambda \in H(f)$ for some $f \in E \cap K$ (see (*)) and $H(f) = fSf$ readily gives $H(f) = x_\lambda Sx_\lambda$. Thus $H(e)$ meets $H(f)$ and, by (*), $H(e) = H(f)$ contrary to the fact that $x_\lambda \in S \setminus H(e)$ and $x_\lambda \in H(f)$.

Remark. With the aid of Lemma 2 and (*) it easily follows that $\mathcal{L}$ is a continuous decomposition of $K$ in this sense: If $\{x_\lambda | \lambda \in \Lambda \}$ is in $K$, if $x_\lambda \rightarrow x$, and if $L_\lambda$ is the member of $\mathcal{L}$ containing $x_\lambda$ then $L_\lambda \rightarrow L$, the member of $\mathcal{L}$ containing $x$. Thus if $\phi: K \rightarrow \mathcal{L}$ is the natural function and if we give $\mathcal{L}$ the usual topology then $\phi$ is a continuous open function and $\mathcal{L}$ is a compact Hausdorff space. Of course we are supposing that $S$ is compact. A similar result is obtained using the sets $H(e)\subseteq E \cap K$.

References


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