MULTIPLY MONOTONE COMPLEX SEQUENCES

W. M. ZARING

1. Introduction. For α real the αth difference of index n, $\Delta^\alpha a_n$, of the real number sequence, $\{a_n\}$, is defined by the equation

$$\Delta^\alpha a_n = \sum_{k=0}^{\infty} \binom{k-\alpha-1}{k} a_{n+k} \tag{1}$$

if the series (1) converges. We say that $\{a_n\}$ is α-monotone for $\alpha \geqq 0$ if

$$\Delta^\alpha a_n \geqq 0; \qquad n = 1, 2, \cdots. \tag{2}$$

This idea of monotonicity was used by E. Jacobsthal [4][1] and K. Knopp [5] for integral α. Later K. Knopp [6] extended the idea to nonintegral values of α. In this paper we shall extend the concept to complex[2] sequences, $\{c_n\}$, in the following manner. Equation (1) suffices to define $\Delta^\alpha c_n$. The inequality (2) we generalize thus.

Definition 1.1. A complex sequence, $\{c_n\}$, will be called (α, ϕ)-monotone when $\alpha \geqq 0$ if

$$|\arg \Delta^\alpha c_n| \leqq \phi < \pi/2, \qquad n = 1, 2, \cdots. \tag{3}$$

Setting $c_n = a_n + ib_n$ it is obvious from equation (1) that $\Delta^\alpha c_n = \Delta^\alpha a_n + i\Delta^\alpha b_n$. Then the inequality (3) is equivalent to the inequality

$$|\Delta^\alpha b_n| \leqq T\Delta^\alpha a_n; \qquad n = 1, 2, \cdots, \tag{4}$$

where $T = \tan\phi$. The inequality (4) is the form of the definition which we will find most useful. In §2 we shall use this definition to extend to sequences of complex numbers with positive real parts the following theorems due to Knopp [6].

Theorem 1.1. *If $\{a_n\}$ is a null sequence, then $\Delta^\alpha a_n$ exists for each index n and for each α such that $\alpha \geqq 0$.*

Theorem 1.2. *If $\{a_n\}$ is a null sequence, then $\{\Delta^\alpha a_n\}$ is a null sequence when $\alpha \geqq 0$.*

Theorem 1.3. *If $\{a_n\}$ is a positive null sequence which is α-monotone, then it is β-monotone if $0 \leqq \beta \leqq \alpha$.*

Presented to the Society, November 27, 1954; received by the editors August 19, 1954 and, in revised form, October 29, 1954.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] I am indebted to A. W. Goodman for suggesting the topic and for the definition here adopted. J. Hjelmslev [3] has generalized the concept of monotonicity in a different direction, producing results which are different from those presented here.

THEOREM 1.4. *If the series* $\sum a_n$ *with positive terms converges and if* $\{a_n\}$ *is* α*-monotone, then for each* β *such that* $0 \leqq \beta \leqq \alpha$ *the differences* $\Delta^{\beta-1}a_n$ *exist and*

$$n^\beta \Delta^{\beta-1} a_n \to 0. \tag{5}$$

THEOREM 1.5. *If* $\sum a_n$ *is a convergent series with positive terms, if* $\{a_n\}$ *is* α*-monotone, and if* $0 \leqq \alpha \leqq 1$, *then*

$$n^\alpha a_n \to 0. \tag{6}$$

2. **Properties of complex sequences.** In this section we shall be concerned with null complex sequences $\{c_n\} \equiv \{a_n + ib_n\}$ which are (α, ϕ)-monotone.

THEOREM 2.1. *If* $\{c_n\}$ *is a null sequence, then* $\Delta^\alpha c_n$ *exists for each index* n *and all* α *such that* $\alpha \geqq 0$.

PROOF. By definition

$$\Delta^\alpha c_n = \sum_{k=0}^{\infty} \binom{k-\alpha-1}{k} c_{n+k}. \tag{7}$$

Since $\{c_n\}$ is bounded and

$$\sum_{k=0}^{\infty} \binom{k-\alpha-1}{k}$$

converges absolutely it follows that the series (7) converges.

THEOREM 2.2. *If* $\{c_n\}$ *is a null sequence and* $\alpha \geqq 0$, *then* $\{\Delta^\alpha c_n\}$ *is a null sequence.*

PROOF. Since $a_n \to 0$ and $b_n \to 0$ we are assured by Theorem 1.2 that $\Delta^\alpha a_n \to 0$ and $\Delta^\alpha b_n \to 0$. Therefore, since $\Delta^\alpha c_n = \Delta^\alpha a_n + i\Delta^\alpha b_n$, it follows that $\Delta^\alpha c_n \to 0$.

THEOREM 2.3. *If* $\{c_n\}$ *is an* (α, ϕ)*-monotone null sequence, then it is* (β, ϕ)*-monotone when* $0 \leqq \beta \leqq \alpha$.

PROOF. For the proof we need two results. First we need

ANDERSON'S THEOREM [1]. *If* $\{c_n\}$ *is a null sequence then*

$$\Delta^s(\Delta^r c_n) = \Delta^{s+r} c_n$$

if $s \geqq -1$, $r \geqq 0$, *and* $s + r \geqq 0$.

Secondly we need the obvious

LEMMA 2.1. *If in the convergent series* $\sum_{k=1}^{\infty} a_k c_k = S \neq 0$, $a_k \geqq 0$, $|\arg c_k| \leqq \phi < \pi/2$, $k = 1, 2, \cdots$, *then* $|\arg S| \leqq \phi$.

The proof of Theorem 2.3 consists of noting that if $\max(\alpha-1,0) \leqq \beta \leqq \alpha$ we may write from Anderson's Theorem, since, $\beta-\alpha \geqq -1$, that

$$\Delta^{\beta} c_n = \Delta^{\beta-\alpha}(\Delta^{\alpha} c_n) = \sum_{k=0}^{\infty} \binom{k+\alpha-\beta-1}{k} \Delta^{\alpha} c_{n+k}.$$

Since by hypothesis $|\arg \Delta^{\alpha} c_n| \leqq \phi$ for all n, and since

$$\binom{k+\alpha-\beta-1}{k} \geqq 0 \qquad \text{for } \alpha \geqq \beta$$

it follows from Lemma 2.1 that $|\arg \Delta^{\beta} c_n| \leqq \phi$ for $\max(\alpha-1, 0) \leqq \beta \leqq \alpha$. Thus if $\alpha-1 \leqq 0$ the proof is complete. If $\alpha-1>0$ we repeat the process where $\max(\alpha-2, 0) \leqq \beta \leqq \alpha-1$. After a finite number of steps we shall reach some number $\alpha-k$ such that $\alpha-k \leqq 0$, $\max(\alpha-k, 0) \leqq \beta \leqq \alpha-k+1$, and $|\arg \Delta^{\beta} c_n| \leqq \phi$. The proof is then complete.

THEOREM 2.4. *If the series* $\sum c_n$ *converges and* $\{c_n\}$ *is* (α, ϕ)-*monotone, then for each* β *such that* $0 \leqq \beta \leqq \alpha$ *the differences* $\Delta^{\beta-1} c_n$ *exist and*

$$n^{\beta} \Delta^{\beta-1} c_n \to 0. \tag{8}$$

PROOF. We consider the cases $\beta \geqq 1$ and $0 \leqq \beta < 1$ separately.

CASE 1. $\beta \geqq 1$. The existence of $\Delta^{\beta-1} c_n$ is assured by Theorem 2.1. Furthermore $\{c_n\}$, being (α, ϕ)-monotone, is $(\beta-1, \phi)$-monotone by Theorem 2.3, since $0 \leqq \beta-1 < \alpha$. Therefore

$$|\Delta^{\beta-1} b_n| \leqq T \Delta^{\beta-1} a_n. \tag{9}$$

On multiplying the inequality (9) by the positive number n^{β} we have

$$|n^{\beta} \Delta^{\beta-1} b_n| \leqq T n^{\beta} \Delta^{\beta-1} a_n. \tag{10}$$

Since $\{a_n\}$ is α-monotone, Theorem 1.4 assures us that $n^{\beta} \Delta^{\beta-1} a_n \to 0$ and therefore, from the inequality (10), $n^{\beta} \Delta^{\beta-1} b_n \to 0$. This establishes that $n^{\beta} \Delta^{\beta-1} c_n \to 0$.

CASE 2. $0 \leqq \beta < 1$. Then

$$\sum_{k=0}^{\infty} \left| \binom{k-\beta}{k} b_{n+k} \right| = \sum_{k=0}^{\infty} \binom{k-\beta}{k} |b_{n+k}| \tag{11}$$

and since $\{c_n\}$ is $(0, \phi)$-monotone, that is since $|b_n| \leqq T a_n$, we may write

$$\sum_{k=0}^{\infty}\left|\binom{k-\beta}{k}b_{n+k}\right| \leqq T\sum_{k=0}^{\infty}\binom{k-\beta}{k}a_{n+k}. \tag{12}$$

Theorem 1.4 assures us that

$$\Delta^{\beta-1}a_n \equiv \sum_{k=0}^{\infty}\binom{k-\beta}{k}a_{n+k}$$

exists. We then have, from the inequality (12), that

$$\Delta^{\beta-1}b_n \equiv \sum_{k=0}^{\infty}\binom{k-\beta}{k}b_{n+k}$$

converges and hence that $\Delta^{\beta-1}c_n$ exists. Since Theorem 1.4 assures us also that $n^\beta\Delta^{\beta-1}a_n \to 0$ we see, from the inequality (12), that $n^\beta\Delta^{\beta-1}b_n \to 0$ and therefore $n^\beta\Delta^{\beta-1}c_n \to 0$, which completes the proof.

THEOREM 2.5. *If $\sum c_n$ is a convergent series and if $\{c_n\}$ is (α, ϕ)-monotone, then for* $0 \leqq \alpha \leqq 1$,

$$n^\alpha c_n \to 0. \tag{13}$$

PROOF. From the convergence of $\sum c_n$ and the condition that $c_n \to 0$ follows the convergence of $\sum a_n$ and the condition that $a_n \to 0$. Since $\{c_n\}$ is (α, ϕ)-monotone it follows that $\{a_n\}$ is α-monotone and hence from Theorem 1.5 that $n^\alpha a_n \to 0$. Since $\{c_n\}$ is (α, ϕ)-monotone, by Theorem 2.3 it is also $(0, \phi)$-monotone and we have that

$$|n^\alpha b_n| \leqq Tn^\alpha a_n \to 0.$$

Therefore $n^\alpha b_n \to 0$, which establishes that $n^\alpha c_n \to 0$.

The restriction $\alpha \leqq 1$ cannot be weakened as the following example shows. Consider $\{1/n^\alpha\}$ where n is a positive integer and α is real. From the following obvious inequality we show that $\{1/n^\alpha\}$ is 2-monotone.

$$\begin{aligned} 0 &\leqq [1/n^{\alpha/2} - 1/(n+2)^{\alpha/2}]^2 \\ &= 1/n^\alpha - 2/[n^{\alpha/2}(n+2)^{\alpha/2}] + 1/(n+2)^\alpha \\ &\leqq 1/n^\alpha - 2/(n+1)^\alpha + 1/(n+2)^\alpha \\ &= \Delta^2(1/n^\alpha). \end{aligned}$$

We then observe that the series $\sum 1/n^\alpha$ with $1 < \alpha \leqq 2$ converges and since $\{1/n^\alpha\}$ is 2-monotone it is α-monotone. But

$$n^\alpha c_n \equiv n^\alpha(1/n^\alpha) = 1.$$

3. Some results of Fejér. The following theorems were proved by Fejér [2] for null real number sequences $\{a_n\}$.

THEOREM 3.1. *If $\{a_n\}$ is a 2-monotone null sequence then $f_1(\theta)$, defined by*

$$f_1(\theta) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos n\theta, \tag{14}$$

is convergent and non-negative in the interval $0<\theta<2\pi$.

THEOREM 3.2. *If $\{a_n\}$ is a 4-monotone null sequence then $f_1(\theta)$, defined by the equation* (14), *is constant or strictly monotone decreasing in the interval* $0<\theta<\pi$.

THEOREM 3.3. *If $\{a_n\}$ is a 2-monotone null sequence, then $f_2(\theta)$, defined by*

$$f_2(\theta) = \sum_{n=1}^{\infty} a_n \sin n\theta, \tag{15}$$

converges and is positive for $0<\theta<\pi$.

THEOREM 3.4. *If, in the equation* (15), *$\{a_n\}$ is a 4-monotone null sequence, then $f_2(\theta)$ is strictly monotone decreasing in the interval* $\pi/2<\theta<\pi$.

THEOREM 3.5. *If $\{a_n\}$ is a 3-monotone null sequence, then $f_3(\theta)$, defined by*

$$f_3(\theta) = \sum_{n=1}^{\infty} a_n \cos (2n-1)\theta, \tag{16}$$

is positive for $0<\theta<\pi/2$, negative for $\pi/2<\theta<\pi$, and is strictly monotone decreasing in the interval $0<\theta<\pi$.

THEOREM 3.6. *If $\{a_n\}$ is a 1-monotone null sequence then $f_4(\theta)$, defined by*

$$f_4(\theta) = \sum_{n=1}^{\infty} a_n \sin (2n-1)\theta, \tag{17}$$

is non-negative for $0<\theta<\pi$.

In the next section we shall extend these theorems to complex sequences. In order to do this certain results which are known [2] will be needed. We prove them here for completeness.

LEMMA 3.1. *If $n \geqq 1$ then,*

$$(18)\quad S_{n1}^{(1)}(\theta) \equiv (n+1)/2 + \sum_{k=1}^{n} (n+1-k)\cos k\theta \geqq 0, \quad 0 \leqq \theta \leqq 2\pi,$$

$$(19)\quad S_{n2}^{(1)}(\theta) \equiv \sum_{k=1}^{n} (n+1-k)\sin k\theta > 0, \quad 0 < \theta < \pi,$$

$$(20)\quad S_{n3}^{(2)}(\theta) \equiv \sum_{k=0}^{n} \binom{n+2-k}{2} \cos(2k+1)\theta > 0, \quad 0 \leqq \theta < \pi/2,$$

$$(21)\quad S_{n4}^{(0)}(\theta) \equiv \sum_{k=1}^{n} \sin(2k-1)\theta > 0, \quad 0 < \theta < \pi.$$

PROOF. The inequality (18) follows immediately from the well known identity

$$(22)\quad S_{n1}^{(1)}(\theta) = \frac{\sin^2[(n+1)/2]\theta}{2\sin^2(\theta/2)}.$$

The equality sign in the inequality (18) can occur in the interior of the interval. However if $p+1$ and $q+1$ are relatively prime, then $S_{p1}^{(1)}(\theta)+S_{q1}^{(1)}(\theta)>0$ for all θ.

To prove the inequality (19) recall first that in any series $\sum a_k$ the νth partial sum of index n, defined by

$$S_n^{(\nu)} = \sum_{k=0}^{n} \binom{n+\nu-k}{\nu} a_k,$$

satisfies the formal identity

$$(23)\quad (1-r)^{-\nu-1} \sum_{k=0}^{\infty} a_k r^k = \sum_{n=0}^{\infty} S_n^{(\nu)} r^n.$$

If we now set

$$(24)\quad F_1(r,\theta) \equiv \frac{(1-r^2)}{(1-2r\cos\theta+r^2)} = 1 + 2\sum_{k=1}^{\infty} r^k \cos k\theta$$

and

$$(25)\quad F_2(r,\theta) \equiv \frac{\sin\theta}{(1-2r\cos\theta+r^2)} = \sum_{k=0}^{\infty} r^k \sin(k+1)\theta,$$

then the identity

$$(26)\quad \frac{1}{(1-r)^2}\,\frac{\sin\theta}{1-r^2}\,F_1(r,\theta) = \frac{1}{(1-r)^2}\,F_2(r,\theta),$$

together with equation (23), implies that $S_{n2}^{(1)}(\theta)$ is a sum of terms

$(\sin\theta)S_{\nu 1}^{(1)}(\theta)$ with positive coefficients, and this establishes the inequality (19).

Next set

$$F_3(r,\theta) \equiv \frac{(1-r)\cos\theta}{1-2r\cos 2\theta+r^2}$$

$$= \sum_{k=0}^{\infty} r^k \cos(2k+1)\theta = \frac{(1-r)\cos\theta}{1-r^2}F_1(r,2\theta). \tag{27}$$

Then

$$\frac{1}{(1-r)^3}F_3(r,\theta) = \frac{\cos\theta}{1-r^2}\,\frac{1}{(1-r)^2}F_1(r,2\theta). \tag{28}$$

Using equation (23) we see that $S_{n3}^{(2)}(\theta)$ is a sum of terms $(\cos\theta)\,S_{n1}^{(1)}(2\theta)$ with positive coefficients, so that the inequality (20) also follows from the inequality (18). Finally if we set

$$F_4(r,\theta) \equiv \frac{(1+r)\sin\theta}{1-2r\cos 2\theta+r^2}$$

$$= \sum_{k=1}^{\infty} r^{k-1}\sin(2k-1)\theta = \frac{\sin\theta}{1-r}F_1(r,2\theta), \tag{29}$$

and multiply both sides by $(1-r)^{-1}$, it is clear that

$$S_{n4}^{(0)}(\theta) = 2(\sin\theta)S_{n-1,1}^{(1)}(2\theta) \qquad \text{if } n \geqq 1.$$

4. Trigonometric series with monotone complex coefficients.

THEOREM 4.1. *If $\{c_n\}$ is a $(2,\phi)$-monotone null sequence, then the series*

$$f_5(\theta) = c_0/2 + \sum_{k=1}^{\infty} c_k \cos k\theta \tag{30}$$

converges and $|\arg f_5(\theta)| \leqq \phi$ for $0<\theta<2\pi$.

PROOF. A necessary and sufficient condition for the convergence of the series (30) is the convergence of the two series

$$a_0/2 + \sum_{k=1}^{\infty} a_k\cos k\theta \quad \text{and} \quad b_0/2 + \sum_{k=1}^{\infty} b_k \cos k\theta. \tag{31}$$

A sufficient condition for the convergence of the series (31) is that their coefficient sequences be null and of bounded variation [8, p. 3]. Since $\{c_n\}$ is $(2,\phi)$-montone, it follows from the definition that

$\{a_n\}$ is 2-monotone and by Theorem 1.3 that $\Delta^1 a_n \equiv a_n - a_{n+1} \geqq 0$. Hence $\sum_{n=0}^{\infty} |\Delta a_n| = a_0$ and the sequence is of bounded variation. It follows, since $\{c_n\}$ being $(2, \phi)$-monotone is $(1, \phi)$-monotone, that

$$\sum_{n=0}^{\infty} |\Delta b_n| \leqq T \sum_{n=0}^{\infty} |\Delta a_n| = T a_0.$$

Therefore $\{b_n\}$ is also of bounded variation, hence the two series (31) converge.

From Abel's partial summation formula applied twice to $c_0/2 + \sum_{k=1}^{n} c_k \cos k\theta$ we obtain

$$c_0/2 + \sum_{k=1}^{n} c_k \cos k\theta = \sum_{k=0}^{n-2} S_{k1}^{(1)}(\theta)\Delta^2 c_k + (c_{n-1} - 2c_n)S_{n-1,1}^{(1)}(\theta) + c_n S_{n1}^{(1)}(\theta). \tag{32}$$

Equation (22) assures us that $S_{n1}^{(1)}(\theta)$ is bounded for a fixed θ such that $0 < \theta < 2\pi$. Since $c_n \to 0$ it then follows from (32), as $n \to \infty$, that

$$f_5(\theta) = \sum_{n=0}^{\infty} S_{n1}^{(1)}(\theta)\Delta^2 c_n.$$

From Lemma 3.1 we have that $S_{n1}^{(1)}(\theta) \geqq 0$ for all θ. Then since $|\arg \Delta^2 c_n| \leqq \phi$ it follows from Lemma 2.1 that $|\arg f_5(\theta)| \leqq \phi$.

THEOREM 4.2. *If, in* (30), $\{c_n\}$ *is a* $(4, \phi)$*-monotone null sequence, then* $f_5(\theta)$ *is either constant or defines a simple curve for* $0 < \theta < \pi$.

PROOF. By Theorem 3.2 the real part of $f_5(\theta)$ is strictly monotone decreasing for $0 < \theta < \pi$ or is constant since $\{a_n\}$ is 4-monotone. Therefore $f_5(\theta_1) = f_5(\theta_2)$ for $0 < \theta_1, \theta_2 < \pi$ implies that the real part of $f_5(\theta)$ is constant or $\theta_1 = \theta_2$. The real part of $f_5(\theta)$ being constant implies that $a_n = 0$ for $n = 1, 2, \cdots$, which implies that $b_n = 0$ for $n = 1, 2, \cdots$, since $\{c_n\}$ is $(0, \phi)$-monotone, which in turn implies that $f_5(\theta)$ is constant.

THEOREM 4.3. *If* $\{c_n\}$ *is a* $(2, \phi)$*-monotone null sequence, then*

$$f_6(\theta) = \sum_{n=1}^{\infty} c_n \sin n\theta \tag{33}$$

converges and $|\arg f_6(\theta)| \leqq \phi$ *for* $0 < \theta < \pi$.

PROOF. The series (33) converges if and only if the two series

$$\sum_{n=1}^{\infty} a_n \sin n\theta \quad \text{and} \quad \sum_{n=1}^{\infty} b_n \sin n\theta \tag{34}$$

converge. Again a sufficient condition for the convergence of the two series (34) is that the coefficient sequences be of bounded variation. In the proof of Theorem 4.1 it was established that $\{a_n\}$ and $\{b_n\}$ are of bounded variation hence the two series (34) converge.

Applying Abel's partial summation formula twice to $\sum_{k=1}^{n} c_k \sin k\theta$ we obtain

$$(35) \qquad \sum_{k=1}^{n} c_k \sin k\theta = \sum_{k=1}^{n-2} S_{k2}^{(1)}(\theta)\Delta^2 c_k + c_{n-1}S_{n-1,2}^{(1)}(\theta) + c_n S_{n5}^{(0)}(\theta)$$

where $S_{n5}^{(0)}(\theta) \equiv \sum_{k=1}^{n} \sin k\theta$. From the well known identity

$$S_{n5}^{(0)}(\theta) \equiv \frac{\cos \theta/2 - \cos (2n+1)\theta/2}{2 \sin \theta/2}$$

we see that $S_{n5}^{(0)}(\theta)$ is bounded for a fixed θ such that $0<\theta<2\pi$ hence, since $c_n \to 0$, $\lim_{n\to\infty} c_n S_{n5}^{(0)}(\theta) = 0$. Since the left member of equation (35) tends to a limit as $n \to \infty$ the right member must also. The first two terms of the right member we can write as

$$\left[\sum_{k=1}^{n-2} S_{k2}^{(1)}(\theta)\Delta^2 a_k + \Delta a_{n-1}S_{n-1,2}^{(1)}(\theta)\right] + i\left[\sum_{k=1}^{n-2} S_{k2}^{(1)}(\theta)\Delta^2 b_k + \Delta b_{n-1}S_{n-1,2}^{(1)}(\theta)\right].$$

Each of the expressions in the brackets must then tend to a limit as $n\to\infty$. Since $\{c_n\}$ is $(2, \phi)$-monotone it follows that $\{a_n\}$ is 2-monotone, that is, $\Delta^2 a_n \geqq 0$ and $\Delta a_n \geqq 0$. By Lemma 3.1 we have $S_{n2}^{(1)}(\theta) > 0$ for $0<\theta<\pi$. We therefore see that the two terms in the first bracket are non-negative. Since their sum is bounded it follows that they must each be bounded. The boundedness of the first term implies the convergence of $\sum_{k=1}^{\infty} S_{k2}^{(1)}(\theta)\Delta^2 a_k$ since it is a sum of positive terms. Since the bracket and the first term tend to a limit it then follows that $\lim_{n\to\infty} \Delta a_n S_{n-1,2}^{(1)}(\theta)$ exists and is non-negative. Since $\{c_n\}$ is $(2, \phi)$-monotone we have that $|\Delta^2 b_n| \leqq T\Delta^2 a_n$ and $\sum_{k=1}^{\infty} S_{k2}^{(1)}(\theta)\Delta^2 b_k$ converges therefore. From this in turn it follows that $\lim_{n\to\infty} \Delta b_n S_{n-1,2}^{(1)}(\theta)$exists. We then obtain from equation (35) as $n\to\infty$

$$f_6(\theta) = \sum_{k=1}^{\infty} S_{k2}^{(1)}(\theta)\Delta^2 c_k + A$$

where $A \equiv \lim_{n\to\infty} \Delta c_{n-1} S_{n-1,2}^{(1)}(\theta)$. Since obviously $|\arg A| \leqq \phi$, and by hypothesis $|\arg \Delta^2 c_n| \leqq \phi$, and since $S_{n2}^{(1)}(\theta)>0$ when $0<\theta<\pi$, it follows from Lemma 2.1 that $|\arg f_6(\theta)| \leqq \phi$ when $0<\theta<\pi$.

THEOREM 4.4. *If, in equation* (33), $\{c_n\}$ *is a* (4, ϕ)*-monotone null sequence, then* $f_6(\theta)$ *is constant or describes a simple curve for* $\pi/2<\theta<\pi$.

PROOF. Since $\{c_n\}$ is (4, ϕ)-monotone we have that $\{a_n\}$ is 4-monotone hence by Theorem 3.4 the real part of $f_6(\theta)$ is strictly monotone decreasing for $\pi/2<\theta<\pi$.

THEOREM 4.5. *If the coefficient sequence,* $\{c_n\}$, *of the series*

$$f_7(\theta) = \sum_{n=0}^{\infty} c_n \cos (2n+1)\theta \tag{36}$$

is a (3, ϕ)*-monotone null sequence, then* $|\arg f_7(\theta)| \leqq \phi$ *for* $0<\theta<\pi/2$ *and* $f_7(\theta)$ *describes a simple curve for* $0<\theta<\pi$.

PROOF. Applying Abel's partial summation formula three times to $\sum_{k=0}^{n} c_k \cos (2k+1)\theta$, we obtain

$$\begin{aligned}\sum_{k=0}^{n} c_k \cos (2k+1)\theta = \sum_{k=0}^{n-3} S_{k3}^{(2)}(\theta)\Delta^3 c_k + \Delta^2 c_{n-2}S_{n-2,3}^{(2)}(\theta) \\ + \Delta c_{n-1}S_{n-1,6}^{(1)}(\theta) + c_n S_{n7}^{(0)}(\theta)\end{aligned} \tag{37}$$

where

$$S_{n-1,6}^{(1)}(\theta) = \sum_{k=0}^{n-1} (n-k) \cos (2k+1)\theta \text{ and } S_{n7}^{(0)}(\theta) = \sum_{k=0}^{n} \cos (2k+1)\theta.$$

It is easily shown that

$$S_{n7}^{(0)}(\theta) = \frac{\sin 2(n+1)\theta}{2 \sin \theta} \quad \text{and} \quad S_{n-1,6}^{(1)}(\theta) = \frac{\cos \theta - \cos (2n+1)\theta}{4 \sin^2 \theta}.$$

From this we see that $S_{n7}^{(0)}(\theta)$ and $S_{n-1,6}^{(1)}(\theta)$ are bounded for a fixed θ such that $0<\theta<\pi$. Then since $c_n \rightarrow 0$ and $\Delta c_n \rightarrow 0$ it follows that the last two terms in the right member of equation (37) tend to zero as $n \rightarrow \infty$. By the same argument employed in the proof of Theorem 4.3 we see that as $n \rightarrow \infty$ we have from equation (37)

$$f_7(\theta) = \sum_{k=0}^{\infty} S_{k3}^{(2)}(\theta)\Delta^3 c_k + B$$

where $B \equiv \lim_{n\to\infty} \Delta^2 c_{n-2} S_{n-2,3}^{(2)}(\theta)$. Since, by Lemma 3.1, $S_{n3}^{(2)}(\theta)>0$ when $0<\theta<\pi/2$, we have that $|\arg B| \leqq \phi$ when $0<\theta<\pi/2$, because by hypothesis $|\arg \Delta^3 c_n| \leqq \phi$. It therefore follows from Lemma 2.1 that $|\arg f_7(\theta)| \leqq \phi$ when $0<\theta<\pi/2$.

Since $\{a_n\}$ is 3-monotone Theorem 3.5 establishes that the real part of $f_7(\theta)$ is strictly monotone decreasing in the interval $0<\theta<\pi$. Therefore $f_7(\theta)$ generates a simple curve for θ in that interval.

THEOREM 4.6. *If* $\{c_n\}$ *in*

$$f_8(\theta) = \sum_{n=1}^{\infty} c_n \sin (2n - 1)\theta \tag{38}$$

is a $(1, \phi)$*-monotone null sequence, then* $|\arg f_8(\theta)| \leqq \phi$ *for* $0<\theta<\pi$.

PROOF. If Abel's partial summation formula is applied to $\sum_{k=1}^{n} c_k \sin (2k-1)\theta$ we obtain

$$\sum_{k=1}^{n} c_k \sin (2k - 1)\theta = \sum_{k=1}^{n-1} S_{k4}^{(0)}(\theta)\Delta c_k + c_n S_{n4}^{(0)}(\theta). \tag{39}$$

By the previous argument since, by Lemma 3.1, $S_{k4}^{(0)}(\theta)>0$ when $0<\theta<\pi$, we obtain as $n\to\infty$

$$f_8(\theta) = \sum_{k=1}^{\infty} S_{k4}^{(0)}(\theta)\Delta c_k + C$$

where $C\equiv\lim_{n\to\infty} c_n S_{n4}^{(0)}(\theta)$. Then since $|\arg C| \leqq \phi$, $|\arg \Delta c_n| \leqq \phi$ and $S_{n4}^{(0)}(\theta)>0$ for $0<\theta<\pi$, we have from Lemma 2.1 that $|\arg f_8(\theta)| \leqq \phi$ when $0<\theta<\pi$.

5. **Remark.** G. Szegö [7] has proved that if $\{a_n\}$ is a 3-monotone sequence of positive real numbers, then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is univalent for $|z|<1$, while if $\{a_n\}$ is merely 2-monotone $f(z)$ need not be univalent. This naturally suggests the problem of finding conditions on (α, ϕ) sufficient to insure the univalence of $\sum_{n=0}^{\infty} c_n z^n$ where $\{c_n\}$ is an (α, ϕ)-monotone sequence of complex numbers.

BIBLIOGRAPHY

1. A. F. Anderson, *Studier over Cesaro's Summabilitetsmetode*, Kopenhagen, 1921.

2. L. Fejér, *Trigonometrische Reihen und Potenzreihen mit mehrfach monotone Koeffizientenfolge*, Trans. Amer. Math. Soc. vol. 39 (1936) pp. 18–59.

3. J. Hjelmslev, *Introduction à la theorie des suites monotones*, Oversigt Over Det Kgl. Danske Videnskabernes Selskabs Forhandlinger, No. 1, 1914.

4. E. Jacobsthal, *Mittelwertbildung und Reihentransformation*, Math. Zeit. vol. 6 (1920) pp. 100–117.

5. K. Knopp, *Mittelwertbildung und Reihentransformation*, Math. Zeit. vol. 6 (1920) pp. 118–123.

6. ———, *Mehrfach Monotone Zahlenfolgen*, Math. Zeit. vol. 22 (1925) pp. 75–85.

7. G. Szegö, *Power series with multiply monotonic sequences of coefficients*, Duke Math. J. vol. 8 (1941) pp. 559–564.

8. A. Zygmund, *Trigonometrical series*, Warsaw, 1935.

UNIVERSITY OF KENTUCKY