A NOTE ON BERNSTEIN'S APPROXIMATION PROBLEM

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This note is intended as an addendum to the paper [2] with which we presuppose familiarity.

Vidav [3] has observed that if condition (3) of [2] is satisfied:

(3) there exists a sequence of polynomials \( p_n \) such that

\[
\lim_{n \to \infty} p_n(u)K(u) = 1, \quad |p_n(u)K(u)| \leq C, \quad -\infty < u < \infty,
\]

then \( \{u^nK(u)\}_0^\infty \) is fundamental in \( C_0(-\infty, \infty) \), unless \( 1/K(z) \) is an entire function.

Unfortunately this criterion is defective in that it leaves in an inconclusive position such functions as \( K(u) = e^{-u^2} \), \( K(u) = \text{sech} \, u \) whose reciprocals are entire, which satisfy the condition (3), and yet which are known to give rise to fundamental sets \( \{u^nK(u)\} \).

In this note we present a sharper criterion which leaves in this indecisive state a smaller class of functions.

**Theorem.** If (3) holds, then \( \{u^nK(u)\} \) is fundamental in \( C_0(-\infty, \infty) \) unless \( 1/K(z) \) is an entire function of exponential type zero.

**Proof.** If (3) holds and \( \{u^nK(u)\} \) is not fundamental, then necessarily [2]

\[
\int_{-\infty}^{\infty} \frac{\log |K(u)|}{1 + u^2} \, du > -\infty.
\]

Let \( F = 1/K \). By virtue of (3) \( K \) does not vanish so that \( F \) is continuous and

\[
\int_{-\infty}^{\infty} \frac{\log |F(u)|}{1 + u^2} \, du < \infty.
\]

Since \( K \) vanishes at \( \pm \infty \), \( |F(u)| > 1 \) for sufficiently large \( u \). Therefore the preceding formula is equivalent to

\[
(A) \quad \int_{-\infty}^{\infty} \frac{\log^+ |F(u)|}{1 + u^2} \, du < \infty.
\]

According to (3) and the principle of harmonic majorization [1, p. 955]

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\[
\log |p_n(z)| \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|y| \log |p_n(\xi)|}{(x-\xi)^2 + y^2} \, d\xi
\]

(B)
\[
\leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|y| \log^+ |F(\xi)|}{(x-\xi)^2 + y^2} \, d\xi + \log C
\]
\[
= U(x, y), \quad y \neq 0, \quad z = x + iy.
\]

The function \(U(x, y)\) is harmonic in \(y > 0\) and in \(y < 0\). As \(x+iy\to x_0, U(x, y)\to U(x_0, 0)\), by a well-known property of the Poisson integral of a continuous function. Hence \(|p_n(z)|\) is majorized by the continuous function \(\exp(U(x, y))\). Since \(p_n(x)\to 1/K(x)\) on the real axis, it follows that \(\lim_{n\to\infty} p_n(z)\) exists for all \(z\) and is an entire function, \(F(z)\). On the real axis \(F(x) = 1/K(x)\), of course. Letting \(n\to\infty\) in (B) gives

(C)
\[
\log |F(z)| \leq U(x, y).
\]

In \(|y| \geq \max (|x|, 2)\),
\[
\frac{1 + \xi^2}{(x-\xi)^2 + y^2} < A.
\]

Also in \(\xi^2 < |z|, |z| > 2\)
\[
\frac{1 + \xi^2}{(x-\xi)^2 + y^2} < \frac{B}{|z|}.
\]

Hence in \(|y| \geq \max (|x|, 2)\),
\[
U(x, y) = \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{\log^+ |F(\xi)|}{1 + \xi^2} \frac{1 + \xi^2}{(x-\xi)^2 + y^2} \, d\xi
\]
\[
= \frac{|y|}{\pi} \int_{|\xi^2 < |x|} + \frac{|y|}{\pi} \int_{|\xi^2 \geq |x|}
\]
\[
< O\left( \frac{|y|}{|z|} \int_{|\xi| < |x|} \frac{\log^+ |F(\xi)|}{1 + \xi^2} \, d\xi \right)
\]
\[
+ O\left( \frac{|y|}{|z|} \int_{|\xi| > |x|} \frac{\log^+ |F(\xi)|}{1 + \xi^2} \, d\xi \right) = o(|y|),
\]

since the integral in the second \(O\)-term is \(o(1)\), by (A). This proves by (C) that \(F(z)\) is of order one, minimum type in \(\pi/4 \leq |\theta| \leq 3\pi/4\). To complete the proof note that
We can now apply a standard Phragmén-Lindelöf argument. The function \( f(z) = p_n(z)e^{-2itn} \) satisfies
\[
|f(z)| < K(e) \quad (|\theta| \leq \pi/4, |z| = r_0(e)),
\]
\[
|f(z)| < 1 \quad (|\theta| = \pi/4; |z| > r_0(e); |\theta| \leq \pi/4, |z| = R),
\]
where \( R \) is any sufficiently large positive number. Hence, by the maximum modulus principle
\[
|f(z)| < K(e),
\]
i.e.
\[
|p_n(z)| < K(e) |e^{itn}| < K(e) e^{1/4}(|\theta| \leq \pi/4, |z| \geq r_0(e)).
\]
It follows that there is a function \( r_1(\eta) \) of \( \eta \) such that for every \( \eta > 0 \)
\[
|p_n(z)| < e^{1/4} \quad (|\theta| \leq \pi/4, |z| \geq r_1(\eta)).
\]
Letting \( n \to \infty \) gives
\[
|F(z)| \leq e^{1/4} \quad (|\theta| \leq \pi/4, |z| \geq r_1(\eta)).
\]
Similarly this inequality can be proved for the sector \( |\pi - \theta| \leq \pi/4 \), which completes the proof of the theorem.

References


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