THE AVERAGE OF THE RECIPROCAL OF A FUNCTION

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1. Introduction. Let \( f(x) \) be an integrable function defined on an interval \( (a, b) \). Its average value is

\[
A(f) = \frac{1}{b - a} \int_a^b f(x) \, dx.
\]

There are circumstances \([1; 2]\) in which one wishes to calculate not \( A(f) \) but

\[
B(f) = [A(f^{-1})]^{-1} = \frac{1}{b - a} \int_a^b \frac{dx}{f(x)},
\]

although \( A(f) \) is much easier to calculate. If \( f(x) \) is constant, \( A(f) = B(f) \), and it is reasonable to expect that \( A(f) \) will be approximately equal to \( B(f) \) if \( f(x) \) does not vary too widely. We propose to determine here the extreme values of the ratio

\[
I(f) = \frac{A(f)}{B(f)}
\]
as \( f \) varies over a special class of functions.

Suppose that \( 0 < \alpha < \beta \), and that \( \mathcal{A} \) is the class of measurable functions \( f(x) \) defined on \( (a, b) \) for which \( \alpha \leq f(x) \leq \beta \). It is a consequence of a result of Pólya and Szegö \([3]\) that

\[
1 \leq I(f) \leq (\alpha + \beta)^2 / 4\alpha\beta
\]
when \( f(x) \) is in \( \mathcal{A} \). If \( \mathcal{B} \) is the class of concave (i.e., arc lies above chord) monotone decreasing functions \( f(x) \) in \( \mathcal{A} \) which assume the values \( \alpha \) and \( \beta \), then we shall prove the better result that

\[
1 \leq I(f) \leq \beta \left[ \frac{\beta \ln(\beta/\alpha)}{\beta - \alpha} - \frac{\beta + \alpha}{2\beta} \right]^2 \frac{1}{2(\beta - \alpha) \left[ \frac{\beta \ln(\beta/\alpha)}{\beta - \alpha} - 1 \right]}
\]
These same bounds apply if \( f(x) \) is a concave, monotone function in \( \mathcal{A} \), since the transformation \( x' = a + b - x \) converts increasing functions into decreasing functions without altering concavity or bounds, and since the right-hand side of (2) is a strictly increasing function of \( \beta/\alpha \).

2. Existence of a maximizing function in \( \mathcal{B} \). Our proof of the ini-

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equality (2) consists of a demonstration of the existence of a function $f_0(x)$ in $\mathfrak{B}$ for which $I(f)$ attains its least upper bound on $\mathfrak{B}$, followed by the deduction of various properties which any such maximizing function must possess. There will be precisely one function in $\mathfrak{B}$ possessing these properties and so it must be the maximizing function.

**Theorem 1.** There exists a function $f_0(x)$ in $\mathfrak{B}$ for which $I(f)$ is a maximum.

We begin by observing that when $f$ is concave, there exists a function $f'(x)$ which is decreasing and integrable on $(a, b)$ such that

$$f(x) = f(a+) + \int_a^x f'(t) dt, \quad a < x < b.$$ 

The set of discontinuities of $f'(x)$ on the open interval $(a, b)$ is a finite or denumerable set $E(f)$ and so $f'(x)$ is the derivative of $f(x)$ except on $E(f)$ and possibly at $a$ and $b$. In addition when $f(x)$ is in $\mathfrak{B}$, $f'(x) \leq 0$. Moreover $f(x)$ is continuous when $a \leq x < b$, and $f(b^-) \geq f(b) = a$.

**Lemma 1.** If $f(x)$ is in $\mathfrak{B}$ and $c = b - M^{-1}(\beta - \alpha) > a$, then $f'(x) \geq -M$ when $a \leq x \leq c$.

For suppose there were a point $y$ such that $a \leq y \leq c$, $f'(y) < -M$. Then, since $f'(x)$ is decreasing, $f'(t) < -M$ when $y \leq t \leq b$. Hence

$$\beta \geq f(y) = f(b-) - \int_y^b f'(t) dt > \alpha + (b - y)M \geq \alpha + (b - c)M = \beta,$$

and this is impossible.

Let $\mu$ be the least upper bound of $I(f)$ on $\mathfrak{B}$, and pick a sequence $f_n(x)$ of functions in $\mathfrak{B}$ for which $\mu = \lim I(f_n)$. These functions may be chosen as continuous. Pick a monotone increasing sequence of numbers $M_k$ such that

$$a < c_k = b - M^{-1}_k(\beta - \alpha) \to b.$$ 

According to Lemma 1, $0 \geq f'_n(x) \geq -M_k$ when $a \leq x \leq c_k$, and so the functions $f_n(x)$ are equicontinuous and uniformly bounded on the closed interval $(a, c_k)$. By Ascoli's theorem, there exists a subsequence of the sequence $f_n(x)$ which converges uniformly to a limit $g_1(x)$ on $(a, c_1)$, a subsequence of this subsequence which converges uniformly to a limit $g_2(x)$ on $(a, c_2)$, etc. It is clear that $g_{k+1}(x) = g_k(x)$ when $a \leq x \leq c_k$ and that if we define $f_0(x)$ as $g_k(x)$ when $a \leq x < b$, then we can by the diagonal process select a subsequence of $f_n(x)$ which
converges uniformly to \( f_0(x) \) on every closed interval \((a, c)\) for which \( c < b \). If we define \( f_0(b) \) to be \( \alpha \), then it is easy to see that \( f_0(x) \) is in \( \mathfrak{B} \) and that

\[
\mu = \lim I(f_n) = I(f_0).
\]

Hence Theorem 1 is true.

Let \( \mathfrak{M} \) be the class of maximizing functions for \( I(f) \) in \( \mathfrak{B} \); then the result of Theorem 1 is that \( \mathfrak{M} \) is not void.

3. **A useful identity.** Most of our remaining analysis will depend in one way or another on the following result, the proof of which is obvious.

**Lemma 2.** If \( f_0(x) \) and \( f(x) \) are in \( \mathfrak{B} \), if \( \eta(x) = f(x) - f_0(x) \), and if \( A(\eta) \neq 0 \), then

\[
\frac{I(f) - I(f_0)}{A(\eta)} = A(f_0^{-1}) - \frac{A(f_0)A(\eta/f_0)}{A(\eta)} - A(\eta/f_0f).
\]

We shall use the lemma first to prove the following result.

**Lemma 3.** If \( f_0(x) \) is in \( \mathfrak{M} \), then \( f_0(x) \) is continuous on the closed interval \((a, b)\).

Since any function in \( \mathfrak{B} \) is continuous when \( a \leq x < b \), it is sufficient to show that \( f_0(x) \) is continuous when \( x = b \). Since \( f_0(b - ) \) exists, suppose that \( f_0(b - ) = \alpha' > \alpha \). Choose a positive number \( \delta \) for which \( \alpha' - \delta > \alpha \). Define \( \eta(x, \varepsilon) \) as 0 when \( a \leq x \leq b - \varepsilon \) and when \( x = b \), and so that \( f(x, \varepsilon) = f_0(x) + \eta(x, \varepsilon) \) is linear on the open interval \((b - \varepsilon, b)\) with limiting end values \( f_0(b - \varepsilon) \) and \( \alpha' - \delta \). Hence

\[
\alpha' - \delta - \beta \leq \eta(x, \varepsilon) < 0 \quad (b - \varepsilon < x < b).
\]

Then \( f(x, \varepsilon) \) is in \( \mathfrak{B} \) for sufficiently small \( \varepsilon \) and so

\[
(I(f) - I(f_0))/A(\eta) \geq 0.
\]

From the first theorem of the mean for integrals,

\[
A(\eta/f_0f) = A(\eta)/f_0(x^*)f(x^*),
\]

in which \( b - \varepsilon < x^* < b \). Since \( A(\eta) \rightarrow 0, f_0(x^*) \rightarrow \alpha' \), and \( f(x^*) \rightarrow \alpha' - \delta \), it follows from Lemma 2 that

\[
A(f_0^{-1}) - A(f_0)/\alpha'(\alpha' - \delta) \geq 0,
\]

\[
\alpha'(\alpha' - \delta) \geq A(f_0)B(f_0).
\]

On the other hand, since \( f_0(x) \) is decreasing, \( f_0(x) \geq \alpha' \) when \( a \leq x < b \),
and so

$$A(f_0)B(f_0) \geq \alpha'^2.$$ 

These last two inequalities are incompatible when $\delta > 0$, and so Lemma 3 must be true.

**4. The behavior of the derivative $f_0'(x)$ of a maximizing function.**

We know that $f_0'(x)$ is a decreasing, nonpositive function and hence the set $E(f_0)$ of its discontinuities on the open set $(a, b)$ is at most denumerable. Our next result is the following lemma.

**Lemma 4.** If $f_0(x)$ is in $\mathcal{W}$ then $f_0'(x)$ is constant on any interval of continuity of $f_0'(x)$.

If Lemma 4 is false, there exists an interval $(y, z)$ such that $f_0'(x)$ is continuous when $y < x < z$ and $f_0'(y+) > f_0'(z-)$. Therefore, there exists a decreasing sequence $x_n$ for which $x_n \rightarrow y$, $f_0'(x_{n+1}) > f_0'(x_n)$. Define $\eta_n(x)$ as 0 when $x$ is not on the interval $(y, x_n)$ and so that $f_n(x) = f_0(x) + \eta_n(x)$ is linear on the interval $(y, x_n)$ with end values $f_0(y)$ and $f_0(x_n)$. If $\eta_n(x)$ were identically zero, $f_0(x)$ would be linear on the interval $(y, x_n)$, and so $f_0'(x_{n+1}) = f_0'(x_n)$. Hence $\eta_n(x)$ does not vanish identically. Moreover, $\eta_n(x) < 0$ when $y < x < x_{n+1}$ since it is convex on the interval $(y, x_n)$ and vanishes at the endpoints of that interval. The function $f_n(x)$ is in $\mathcal{B}$ and so

$$\lim_{n \rightarrow \infty} \frac{I(f_n) - I(f_0)}{A(\eta_n)} \geq 0.$$ 

Moreover,

$$A(\eta_n/f_0f_n) = \frac{A(\eta_n)}{f_0(x_*)f_n(x_*)},$$

in which $y < x_* < x_n$, $A(\eta_n) \rightarrow 0$, and so

$$A(f_0^{-1}) - A(f_0)/[f_0(y)]^2 \geq 0,$$

(3) $$[f_0(y)]^2 \geq A(f_0)B(f_0) \equiv C^2.$$ 

Now define $\xi_n(x)$ as 0 when $x$ is not in the interval $(y, x_n)$, and so that $g_n(x) = f_0(x) + \xi_n(x)$ is linear on an interval $(y, \gamma_n)$ with slope $f_0'(y+)$, is linear on the interval $(\gamma_n, x_n)$ with slope $f_0'(x_n)$, and is continuous on $(a, b)$. Then $y < \gamma_n < x_n$ and $\xi_n(\gamma_n) > 0$, since $f_0(x)$ is not linear on the interval $(y, x_n)$. Moreover $\xi_n(x) \geq 0$ since $f_0(x)$ is concave and hence $\xi_n(x) > 0$ when $y < x < x_n$. The function $g_n(x)$ is in $\mathcal{B}$ and so
This limit is evaluated exactly as in the preceding paragraph, and we deduce that

\[ [f_0(y)]^2 \leq A(f_0)B(f_0) \equiv C^2. \]

We conclude from the inequalities (3) and (4) that \( f_0(y) = C \). It is clear that we could use the same arguments at \( z \) and deduce also that \( f_0(z) = C \). Since \( f_0(x) \) is decreasing, we must then have that \( f_0(x) = C \) when \( y \leq x \leq z \), and so \( f_0'(x) = 0 = f_0'(y+) = f_0'(z-) \). From this contradiction we infer the truth of Lemma 4.

5. **The set \( E(f_0) \) of discontinuities of \( f_0'(x) \).** We are going to show ultimately that the set \( E(f_0) \) consists of precisely one point for any function \( f_0(x) \) in \( \mathfrak{M} \). The first step in this demonstration is the following result.

**Lemma 5.** If \( f_0(x) \) is in \( \mathfrak{M} \), then the set \( E(f_0) \) is not void.

If \( E(f_0) \) is void, then \((a, b)\) is an interval of continuity of \( f_0'(x) \) and so \( f_0'(x) \) is constant on \((a, b)\). This constant must be

\[ \frac{(\beta - \alpha)(x - a)}{(b - a)}, \]

and so

\[ f_0(x) = \beta - \frac{(\beta - \alpha)(x - a)}{(b - a)}, \]

\[ I(f_0) = \frac{(\alpha + \beta) \ln (\beta/\alpha)}{2(\beta - \alpha)}. \]

Let us define \( f(x, \epsilon) \) so that

\[ f(x, \epsilon) = \begin{cases} \beta & \text{if } a \leq x \leq a + \epsilon, \\ \beta - (\beta - \alpha)(x - a - \epsilon)/(b - a - \epsilon) & \text{if } a + \epsilon \leq x \leq b. \end{cases} \]

Then \( f(x, \epsilon) \) is in \( \mathfrak{B} \) and

\[ I(f) = I(f_0) + \frac{\epsilon}{2(b - a)} \left( \frac{1 + x}{\xi} - \frac{2 \ln \xi}{\xi - 1} \right), \]

\[ + \frac{\epsilon^2}{2(b - a)^2} \left( \frac{\xi - 1}{\xi} - \ln \xi \right), \]

in which \( \xi = \beta/\alpha > 1 \). It is easy to verify that the coefficient of
\(\epsilon/2(b-a)\) is positive when \(\xi>1\) and hence \(I(f)>I(f_0)\) for sufficiently small positive \(\epsilon\). From this contradiction we infer the truth of Lemma 5.

**Lemma 6.** If \(f_0(x)\) is in \(M\) and if \(y\) is any point in the set \(E(f_0)\) then

\[
(6) \quad f_0(y) \geq C = \left\{ A(f_0)B(f_0) \right\}^{1/2}.
\]

We define \(\eta(x, \epsilon)\) as 0 when \(|x-y| \geq \epsilon\) and so that \(f(x, \epsilon) = f_0(x) + \eta(x, \epsilon)\) is linear on the interval \((y-\epsilon, y+\epsilon)\) and continuous on \((a, b)\).

Then \(\eta(x, \epsilon) < 0\) when \(y-\epsilon < x < y+\epsilon\), \(f(x, \epsilon)\) is in \(A\), and so

\[
\lim_{\epsilon \to 0} \frac{I(f) - I(f_0)}{A(\eta)} \geq 0.
\]

The limit may be evaluated exactly as in the proof of Lemma 4, and leads immediately to the inequality (6).

**Lemma 7.** If \(f_0(x)\) is in \(M\) and if \(y\) is any point in the set \(E(f_0)\) for which there exists another point \(z\) in \(E(f_0)\) such that \(z<y\) and the interval \((z, y)\) contains no other points of \(E(f_0)\), then

\[
(7) \quad f_0(y) = C = \left\{ A(f_0)B(f_0) \right\}^{1/2}.
\]

Choose \(\lambda\) so that \(f'_0(z+) < \lambda < f'_0(z-)\). Define \(\eta(x, \lambda)\) as 0 when \(a \leq x \leq z\) and when \(y \leq x \leq b\) and so that \(f(x, \lambda) = f_0(x) + \eta(x, \lambda)\) is linear on an interval \((z, \gamma)\) with slope \(\lambda\), is linear on the interval \((\gamma, y)\) with slope \(f'_0(y+)\), and is continuous on \((a, b)\). Since \(f_0(x)\) is linear on the interval \((z, \gamma)\) with slope \(f'_0(z+)\), we have that \(z<\gamma<y\) and that \(\eta(x, \lambda)>0\) when \(z<x<y\). Hence

\[
\lim_{\lambda \to f'_0(z+)} \sup_{\lambda} \frac{I(f) - I(f_0)}{A(\eta)} \leq 0.
\]

Since \(\eta(x, \lambda) \geq 0\) and both \(f_0\) and \(f\) are decreasing

\[-A(\eta/f_0) \geq -A(\eta)/[f_0(y)]^2.
\]

Since \(A(\eta) \to 0\), we deduce that

\[
0 \geq A(f_0^{-1}) - A(f_0)/[f_0(y)]^2,
\]

\[
f_0(y) \leq C.
\]

Since Lemma 6 holds, we conclude that Lemma 7 is true.

**Lemma 8.** If \(f_0(x)\) maximizes \(I(f)\) on \(A\), and if \(y\) is any point in the set \(E(f_0)\) for which there exists an increasing sequence \(z_n\) of points of \(E(f_0)\) which converge to \(y\), then \(f_0(y) = C = \left\{ A(f_0)B(f_0) \right\}^{1/2}.

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Define \( \eta_n(x) \) as 0 when \( a \leq x \leq z_n \) and \( y \leq x \leq b \), and so that \( f_n(x) = f_0(x) + \eta_n(x) \) is linear on an interval \( (z_n, \gamma_n) \) with slope \( f'_0(z_n^-) \), is linear on the interval \( (\gamma_n, y) \) with slope \( f'_0(y^+) \), and is continuous on \( (a, b) \). Then \( z_n < \gamma_n < y \) and \( \eta_n(x) > 0 \) when \( z_n < x < y \) since a corner \( z_{n+1} \) occurs on the open interval \( (z_n, y) \). Hence

\[
A(f_0^{-1}) - \frac{A(f_0)}{[f_0(y)]^2} = \lim_{n \to \infty} \frac{I(f_n) - I(f_0)}{A(\eta_n)} \leq 0,
\]

\[f_0(y) \leq C.
\]

Since Lemma 6 holds, we conclude that Lemma 8 is true.

**Lemma 9.** If \( f_0(x) \) is in \( \mathcal{M} \), then the set \( E(f_0) \) has at most two points.

Suppose on the contrary that \( E(f_0) \) has three distinct points \( u < v < w \). Then either Lemma 7 or Lemma 8 applies to the points \( v \) and \( w \) and so \( f_0(v) = f_0(w) = C \). Since \( f_0(x) \) is decreasing, \( f_0(x) = C \) when \( v < x < w \), \( f'_0(v^+) = 0 \). On the other hand, \( f'_0(x) \) is a nonpositive decreasing function and \( f'_0(v^+) < f'_0(v^-) \leq 0 \). From this contradiction we infer that Lemma 9 is true.

**Lemma 10.** If \( f_0(x) \) is in \( \mathcal{M} \), and if the set \( E(f_0) \) has exactly two points \( y < z \), then \( f_0(y) = \beta \), \( f_0(z) = C \). If \( E \) has exactly one point \( y \), then either \( f_0(y) = \beta \) or \( f_0(y) = C = \{ A(f_0) B(f_0) \}^{1/2} \).

Suppose in either case that \( f_0(y) < \beta \). Then define \( \eta(x, \epsilon) \) as 0 when \( x \geq y \) and so that \( f(x, \epsilon) = f_0(x) + \eta(x, \epsilon) \) is linear on the interval \( (y - \epsilon, y) \) with slope \( f'_0(y^+) \), is linear on the interval \( (a, y - \epsilon) \), assumes the value \( \beta \) when \( x = a \), and is continuous on \( (a, b) \). Then \( \eta(x, \epsilon) > 0 \) when \( 0 < x < y \), \( A(\eta) \to 0 \) with \( \epsilon \), and \( f(x, \epsilon) \) is in \( \mathcal{B} \) for sufficiently small positive \( \epsilon \). Hence

\[-A(\eta/f_0) \geq -A(\eta)/[f_0(y)]^2.
\]

Since \( A(\eta) \to 0 \),

\[0 \geq \lim \sup \frac{I(f) - I(f_0)}{A(\eta)} \geq A(f_0^{-1}) - \frac{A(f_0)}{[f_0(y)]^2},
\]

and so \( f_0(y) \leq C \). Since Lemma 6 holds, \( f_0(y) = C \). This is sufficient to prove the second sentence of Lemma 10. If \( E(f_0) \) has another point \( z > y \), then \( f_0(z) = C \) also since Lemma 7 holds. Hence \( f_0(x) = C \) when \( y \leq x \leq z \), \( f'_0(y^+) = 0 \), and this is impossible. From this contradiction we infer the truth of the first sentence of Lemma 10.

**Lemma 11.** If \( f_0(x) \) is in \( \mathcal{M} \), then the set \( E(f_0) \) has exactly one point.
Suppose on the contrary that \( E(f_0) \) has exactly two distinct points \( y < z \). Then from Lemma 10, \( f_0(y) = \beta, f_0(z) = C \) and \( f_0(x) \) is linear on the intervals \((a, y), (y, z)\) and \((z, b)\). Define \( f(x, \epsilon) \) to be \( f_0(x) \) on the intervals \((a, y)\) and \((z, b)\), linear with slope \( f_0'(z+) \) on the interval \((z - \epsilon, z)\), linear on the interval \((y, z - \epsilon)\), and continuous on \((a, b)\). Define

\[
\Delta = \frac{(z - y)}{2(b - a)} \left[ f_0'(z-) - f_0'(z+) \right].
\]

Then \( \Delta > 0 \), and

\[
A(f) = A(f_0') + \epsilon \Delta,
A(f^{-1}) = A(f_0^{-1}) + \epsilon \Delta \Delta_1 + O(\epsilon^2)
\]
in which

\[
\Delta_1 = \frac{2}{\beta - C} \left\{ \frac{\ln(\beta/C)}{\beta - C} - 1 \right\}.
\]

Hence

\[
I(f) = I(f_0) + \epsilon \Delta \{ A(f_0^{-1}) + A(f_0)\Delta_1 \} + O(\epsilon^2),
\]
and so the coefficient of \( \epsilon \Delta \) must be nonpositive. On the other hand, since \( C^2 = A(f_0)B(f_0) \),

\[
A(f_0^{-1}) + A(f_0)\Delta_1 = \frac{A(f_0)}{C^2} \left\{ 1 + \frac{2}{\xi - 1} \left( \frac{\ln \xi}{\xi - 1} - 1 \right) \right\},
\]
in which \( \xi = \beta/C > 1 \). The quantity within the braces is always positive when \( \xi > 1 \), and from this contradiction we deduce the truth of Lemma 11.

**Lemma 12.** If \( f_0(x) \) is in \( \mathbb{M} \) the value \( f_0(y) \) at the unique discontinuity \( y \) of \( f_0(x) \) is \( \beta \).

Suppose on the contrary that \( f_0(y) < \beta \). According to Lemma 10, \( f_0(y) = C \). Let \( f(x, \epsilon) \) be defined as in the proof of Lemma 10. Then

\[
A(f) = A(f_0) + \epsilon \Delta_2,
A(f^{-1}) = A(f_0^{-1}) + \epsilon \Delta_2 \Delta_1 + O(\epsilon^2),
\]
in which

\[
\Delta_2 = \frac{(y - a)}{2(b - a)} \left[ f_0'(y-) - f_0'(y+) \right] > 0.
\]
Hence
\[ I(f) = I(f_0) + \epsilon \Delta_2 \left\{ A(f_0^{-1}) + A(f_0) \Delta_1 \right\} + O(\epsilon^2). \]

Since the coefficient of \( \epsilon \Delta_2 \) is the same quantity encountered in the proof of Lemma 11, it is positive and so \( I(f) > I(f_0) \) for sufficiently small positive \( \epsilon \). From this contradiction we infer the truth of Lemma 12.

6. The maximizing function \( f_0(x) \). We are now in a position to prove our principal result.

**Theorem 2.** If \( f(x) \) is in \( B \), then
\[ 1 \leq I(f) \leq I(f_0), \]
in which
\[ f_0(x) = \begin{cases} \beta & (a \leq x \leq y), \\ \beta - (\beta - \alpha)(x - y)/(b - y) & (y \leq x \leq b), \end{cases} \]
\[ y = a + \frac{(b - a)((1 + \xi)/\xi - 2 \ln \xi/(\xi - 1))}{2(\ln \xi - (\xi - 1)/\xi)}. \]

As a consequence of the preceding lemmas, we know that any maximizing function \( f_0(x) \) must be of the form described in Theorem 2 for some value \( y \). If \( y = a + \epsilon \) we have already \( I(f_0) \) in equation (5).

This equation may be written as
\[ I(f_0) = L + Mu + Nu^2, \]
in which
\[ L = \frac{(\xi + 1) \ln \xi}{2(\xi - 1)}, \]
\[ M = \frac{1}{2} \left( \frac{1 + \xi}{\xi} - \frac{2 \ln \xi}{\xi - 1} \right) > 0, \]
\[ N = \frac{1}{2} \left( \frac{\xi - 1}{\xi} - \ln \xi \right) < 0, \]
\[ u = (y - a)/(b - a), \]
\[ \xi = \beta/\alpha. \]

This function of \( u \) is concave and attains its maximum value when
\[ u = -M/2N. \]

Since \( M > 0 \), \( N < 0 \) and \( 2N + M < 0 \), this value of \( u \) lies in the open
interval \((0, 1)\) and yields the value of \(y\) specified in the theorem. The maximum value is \((4LN - M^2)/4N\), and this value is the one specified in the inequality (2).

Hence the only maximizing function possible is the one specified in Theorem 2. Since a maximizing function is known to exist, this function must be a maximizing function and in fact the only maximizing function.

A short table of values of the upper bounds in (1) and (2) as a function of the ratio \(\beta/\alpha\) is given below.

<table>
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<th>(\beta/\alpha)</th>
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References


Nuclear Development Associates, Inc.