ON CERTAIN SUBSETS OF FINITE BOOLEAN ALGEBRAS

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1. The boolean algebra $B_n$, of finite dimension $n$, may be written as the direct union $B_1 \times B_1 \times \cdots \times B_1$ of $n$ copies of $B_1$. Consequently each element $u$ of $B_n$ may be represented by an $n$-digit binary number. Let $G_n$ be the group of those permutations on the elements of $B_n$ which interchange or invert various of the factors in the direct union expansion. Thus the elements of $G_n$ permute the components of the $u$ or interchange 0, 1 in certain components of every $u$ in $B_n$. The order of $G_n$ is therefore $2^n n!$.

Two subsets of $B_n$ will be called congruent modulo $G_n$ if one is the image of the other under transformation by an element of $G_n$. Clearly sets congruent modulo $G_n$ have the same number of elements. The number of elements in a subset will be called the order of the subset. Let $N_n(s)$ be the number of congruence classes of subsets of order $s$. Note that $N_n(s) = N_n(2^n - s)$. Pólya [1] has calculated $N_n(s)$ for $0 \leq s \leq 2^n$ and $n = 1, 2, 3, 4$, and Slepian [2] has found the values of $N_n = \sum_{i=0}^{2^n} N_n^{(i)}$ for $n = 5, 6$. Trivially, $N_n(0) = N_n(1) = 1$, all $n$; and it is almost as obvious that $N_n^{(2)} = n$, all $n \geq 1$ (see §2).

In this note an elementary argument is given which yields an explicit expression for $N_n^{(3)}$ good for all $n > 1$ (Theorem 2).

2. The procedure for calculating $N_n^{(3)}$ is based on the notion of the “dimension” of a subset of $B_n$. Let $S$ be a subset of $B_n$ whose order is at least 2. Let $\vee S$ and $\wedge S$ be the lattice-union and lattice-intersection, respectively, of the elements of $S$. (The symbols $\vee$, $\wedge$, and $\triangleq$ will be used for the lattice operations and the ordering relation in $B_n$, while $\cup$, $\cap$, and $\subseteq$ will be reserved for their set-theoretical counterparts.) Since the quotient $\vee S / \wedge S$ is relatively complemented, it is a boolean algebra, say $B_r$ if of dimension $r$. $B_r$ may be called the connected closure of $S$. The relation between a subset $S$ and its connected closure will be written: $S < B_r$. The dimension of $S$ is defined to be the dimension of $B_r$ if $S < B_r$. Thus $S < B_r$ implies $\dim S = r$. If the order of $S$ is two, $\dim S$ is simply the usual metric in $B_n$.

The technique to be used for counting incongruent sets in $B_n$ is to determine the number of incongruent sets of given order having maximal dimension in $B_k$ for each $k \leq n$. Thus, for instance, $N_n^{(2)} = n$,
all \( n \geq 1 \); for in \( B_k \) all sets of order two and maximal dimension are congruent.

**Lemma.** Suppose \( S \) is a subset of \( B_n \) of order 3 or more. If \( S = \{ u \} \cup T \) and \( T \leq B_r \), then \( S \) has maximal dimension if and only if \( u \in B'_r \) where \( B'_r \) is the set of the complements of the elements of \( B_r \).

**Proof.** If \( u \in B'_r \), then \( u \cap (\cup T) \leq u \cap u' = 0 \) and \( u \cup (\cup T) \geq u \cup u' = I \), i.e. \( S = \{ u \} \cup T \) has maximal dimension. But if \( u \in B'_r \), and \( v \in B_r \) then either \( u \cap v > 0 \) or \( u \cup v < I \), since complements are unique, and in this case \( \dim S \) cannot be maximal.

3. Consider now the case \( s = 3 \) and suppose \( S = \{ u_1, u_2, u_3 \} \). Let \( r_i = \dim \{ u_j, u_k \} \), where \( i, j, k \) is some permutation of \( 1, 2, 3 \). Clearly \( 1 \leq r_i \leq n \).

**Theorem 1.** \( \dim \{ u_1, u_2, u_3 \} = n \) if and only if \( r_1 + r_2 + r_3 = 2n \).

**Proof.** If \( u_i = x^i_1 x^i_2 \cdots x^i_n, 1 \leq i \leq 3 \), are the binary representations of the \( u_i \), no generality is lost by assuming \( x^1_i = 1, 1 \leq i \leq n; x^2_i = 1, 1 \leq i \leq n-r_3; x^3_i = 0, n-r_3+1 \leq i \leq n. \) Suppose further that \( x^3_i = 0, 1 \leq i \leq k \) and \( n-r_3+1 \leq i \leq n-l \); \( x^3_i = 1, k+1 \leq i \leq n-r_3 \) and \( n-l+1 \leq i \leq n \); where \( k \leq n-r_3, l \leq r_3 \). The lemma implies that \( \dim \{ u_1, u_2, u_3 \} = n \) if and only if \( k = n-r_3 \). But \( r_1 = k+l, r_2 = k+r_3-l, \) so that \( r_1 + r_2 + r_3 = 2(k+r_3) \); from which the theorem follows.

The expressions for the binary components also give the following immediate corollary.

**Corollary.** If \( \{ u_1, u_2, u_3 \}, \{ v_1, v_2, v_3 \} \) are subsets of \( B_n \) with maximal dimension, and if

\[
\dim \{ u_i, u_j \} = \dim \{ v_i, v_j \}, \quad 1 \leq i, j \leq 3,
\]

then \( \{ u_1, u_2, u_3 \} \) and \( \{ v_1, v_2, v_3 \} \) are congruent modulo \( G_n \).

Theorem 1 and the corollary imply that the number of incongruent sets of order 3 having maximal dimension in \( B_n \) is precisely the number of solutions of the following diophantine system:

\[
2n = x + y + z, \quad 1 \leq x \leq y \leq z \leq n.
\]

**Theorem 2.**

\[
N^{(3)}_n = \sum_{k=2}^{n} \left\{ \left[ \frac{k}{3} \right] + \sum_{r=0}^{[k/3]} \left[ \frac{k - 3r}{2} \right] \right\}.
\]

**Proof.** For each \( k, 2 \leq k \leq n \), it suffices to count the number of solutions, \( N^{(3)}_k - N^{(3)}_{k-1} \), of
2k = x + y + z, \quad 1 \leq x \leq y \leq z \leq k.

Suppose \( z = k - r \), then \( 0 \leq r \leq \lfloor k/3 \rfloor \) since \( z = 2k - x - y \geq 2k - 2z \), or \( 3z \geq 2k \). Now \( y \leq z = k - r \) and \( x = k + r - y \geq k + r - (k - r) = 2r \), but \( x \leq (k + r)/2 \) so \( 2r \leq x \leq (k + r)/2 \). For \( r = 0 \), there are \( \lfloor k/2 \rfloor \) possible values for \( x \); and if \( r > 0 \), there are \( \lfloor (k + r)/2 - (2r - 1) \rfloor \). Hence

\[
N^{(3)}_k - N^{(3)}_{k-1} = \left\lceil \frac{k}{2} \right\rceil + \sum_{r=1}^{\lfloor k/3 \rfloor} \left\lceil \frac{k - 3r + 2}{2} \right\rceil = -1 + \sum_{r=0}^{\lfloor k/3 \rfloor} \left\lceil \frac{k - 3r + 2}{2} \right\rceil = \left\lceil \frac{k}{3} \right\rceil + \sum_{r=0}^{\lfloor k/3 \rfloor} \left\lceil \frac{k - 3r}{2} \right\rceil.
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REFERENCES


Bell Telephone Laboratories and
The Rice Institute