ON \( u \)-STABLE COMMUTATIVE POWER-ASSOCIATIVE ALGEBRAS

LOUIS A. KOKORIS

A commutative power-associative algebra \( A \) of characteristic > 5 with an idempotent \( u \) may be written\(^1\) as the supplementary sum \( A = A_u(1) + A_u(1/2) + A_u(0) \) where \( A_u(\lambda) \) is the set of all \( x_\lambda \) in \( A \) with the property \( x_\lambda u = \lambda x_\lambda \). The subspaces \( A_u(1) \) and \( A_u(0) \) are orthogonal subalgebras, \( [A_u(1/2)]^2 \subseteq A_u(1) + A_u(0) \) and \( A_u(\lambda)A_u(1/2) \subseteq A_u(1/2) + A_u(1 - \lambda) \) for \( \lambda = 0, 1 \). The algebra \( A \) is called \( u \)-stable if \( A_u(\lambda)A_u(1/2) \subseteq A_u(1/2) \) and is called stable if it is \( u \)-stable for every idempotent element \( u \) of \( A \).

A. A. Albert has shown in [3] that a simple commutative power-associative algebra \( A \) of degree > 1 over its center \( F \) with characteristic prime to 30 is a Jordan algebra if and only if it is stable. Moreover, it is known that every simple algebra of degree > 2 is a Jordan algebra. Thus there remains the problem of determining the nonstable simple algebras of degree two. There do exist simple algebras of characteristic \( p > 5 \) which are not Jordan algebras [3; 4]. Of course, these algebras are not stable, although they may be \( u \)-stable for some idempotent \( u \). In this paper we shall obtain the following result.

**Theorem.** Let \( A \) be a \( u \)-stable simple commutative power-associative algebra of degree 2 over its center \( F \) of characteristic zero. Then \( A \) is a Jordan algebra.

We shall use all of the results of [3] so we shall adopt the notations of that paper. In particular, all the results of the section giving properties of \( u \)-stable algebras will be used. For convenience let us state a few of the required results here.

In a simple \( u \)-stable algebra \( A \) there exists an element \( w \) in \( A_u(1/2) \) such that \( w^2 = 1 \). Then \( A_u(1) = uB, A_u(0) = vB, \) and \( A_u(1/2) = wB + G \) where \( B \) is the set of all elements \( b \) of \( C = A_u(1) + A_u(0) \) with the property \((wb)w = b\) and \( G \) is the set of all quantities \( g \) of \( A_u(1/2) \) with the property \( wg = 0 \). Since \( e = (1/2)(1 + w) \) and \( f = 1 - e \) are orthogonal idempotents, we may decompose \( A \) relative to \( e \). It can be shown that \( A_e(1) = eB, A_e(0) = fB, \) and \( A_e(1/2) = B(u - v) + G \). The set \( B \) is a subalgebra of \( C \) and the product of two elements in \( G \) is in \( B \). Also,

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\(^1\) The results of this paragraph are given in [1]. The numbers in brackets refer to the bibliography at the end of the paper.
the following multiplicative relationships exist for any \( a, b \) in \( B \), \( g \) in \( G \).

(1) \( w(bu) = w(bv) = \frac{1}{2} w(b) \),

(2) \( (wa)b = w(ab), (wa)(wb) = ab \),

(3) \( g[b(u - v)] = wd \),

(4) \( gb = h - wc \),

(5) \( (wb)g + w(gb) = -d(u - v) \),

(6) \( (wb)[a(u - v)] = k \),

for \( h, k \) in \( G \), and \( c, d \) in \( B \). The quantity \( d \) in relation (5) is the \( d \) of (3).

The theorem can evidently be reduced to the case where \( F \) is algebraically closed. Then\(^2\) \( \text{Au}(1) = uF + N_1 \) and \( \text{Au}(0) = vF + N_0 \) where \( N_1 \) is the radical of \( A_u(\lambda) \) and \( N' = N = N + N(u - v) \) is the radical of \( C \) where \( N \) is the radical of \( B \). Similarly, \( A_e(1) = eF + M_1 \), \( A_e(0) = fF + M_0 \), \( M_1 \) is the set of all elements \( ec \) where \( c \) is in \( N \) and we have the corresponding result for \( M_0 \).

The following important known\(^3\) lemma can now be stated.

**Lemma 1.** Let \( A \) be a commutative power-associative algebra of degree two over a field \( F \) of characteristic zero. Then \( A_e(1/2)A_e(1) \subseteq A_e(1/2) + M_0 \) and \( A_e(1/2)A_e(0) \subseteq A_e(1/2) + M_1 \). Note that the result of the lemma is not vacuous here since we are assuming \( u \)-stability only.

Consider the product \((eB)G\) which was used to obtain (4) and (5). By Lemma 1, \((eB)G \subseteq A_e(1/2) + M_0\) so that \((b + wb)g = a(u - v) + h + c - wc\) for \( a, b \) in \( B \), \( g, h \) in \( G \), and \( c \) in \( N \) the radical of \( B \). Then \((wb)g = a(u - v) + c\) and it is shown in [3] that \( a = -d \) of relation (3). Also\(^4\) the quantity \( d \) in (3) and (5) is in \( N \). These results may be stated as follows.

**Lemma 2.** Let \( A \) be a \( u \)-stable commutative power-associative algebra over a field of characteristic zero. Then \( GB \subseteq G + wN, G[B(u - v)] \subseteq wN, w(GB) \subseteq N, (wB)G \subseteq N' \), and \( w(GB) + (wB)G \subseteq N(u - v) \).

It will also be necessary to have

**Lemma 3.** The product \( G\{ (wB)[B(u - v)] \} \subseteq N \).

For proof substitute \( x = g, y = a, z = b(u - v) \) into the multilinear

\(^2\) By Theorem 2 of [2].

\(^3\) See Theorem 6 of [5].

\(^4\) [2, Lemma 10].
identity obtained from the associativity of fourth powers. Relation (1) implies \( wz = w(az) = 0 \) and we have \( wg = 0 \) by definition of \( G \). Thus

\[
4(wa)(gz) = w[(ga)z + (gz)a + g(az)] + g[(wa)z] + a[(gz)w] + z[(wa)g + w(ga)].
\]

By (3) and (2), \((wa)(gz)\) is in \((wB)(wN) \subseteq N\). The quantity \( ga \) is in \( G+wN \) by (4); hence \((ga)z\) is in \( G[B(u-v)]+(wN)[B(u-v)] \). Consequently, (3) and (6) imply \( w[(ga)z] \) in \( N \). Since \((gz)a\) lies in \( \{G[B(u-v)]\}B \subseteq (wN)B \subseteq wN \), \( w[(gz)a] \) is in \( N \). Also \( w[g(az)] \) is in \( w \cdot G[B(u-v)] \subseteq w(wN) = N \). The product \( a[(gz)w] \) is in \( N \) and \( z[(wa)g + w(gz)] \) is contained in \( [B(u-v)] \cdot [N(u-v)] \subseteq N \). This completes the proof of Lemma 3.

The proofs of Lemmas 15 and 17 of [3] which state that \([Au(1/2) \cdot N']C \subseteq N'A_u(1/2) \) and \([Au(1/2)N']A_u(1/2) \subseteq N' \) follow without change. We also have without change that \( N' + A_u(1/2)N' \) is an ideal of \( A \). Since \( A \) is simple, this ideal must be zero because it does not contain the identity element. Thus \( A = uF + vF + A_u(1/2) \), which is a Jordan algebra. A Jordan algebra is stable so we have as a corollary that a simple commutative power-associative algebra of degree 2 and characteristic 0 is stable if and only if it is \( u \)-stable.

**Bibliography**


**University of Washington**

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\(^5\) The identity is stated in all of our references.