ON SPACES FILLED UP BY CONTINUOUS COLLECTIONS
OF ATRIODIC CONTINUOUS CURVES

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Throughout this paper, \( G \) will denote a nondegenerate continuous collection of atriodic continuous curves (i.e. arcs or simple closed curves) filling up a compact metric continuum \( M \). As is well known, we may regard \( G \) itself as a compact metric continuum, with the elements of the collection \( G \) as the points of the space \( G \) and with \( G \), as a space, the image of \( M \) under an open continuous mapping whose inverse sets are the elements of the collection \( G \).

The principal results of this paper are the following theorems.

**Theorem I.** No closed, totally disconnected point set separates \( M \).\(^1\)

**Theorem II.** If each element of \( G \) is an arc, then no closed, totally disconnected point set separates any connected open subset of \( M \).

**Theorem III.** If \( G \) is a two-dimensional Cantor manifold,\(^2\) then \( M \) is not separated by any rational curve.\(^3\)

As an immediate corollary of Theorem I, we have

**Theorem IV.**\(^4\) At no point of \( M \) is the dimension of \( M \) less than 2.

A well known result cited in [2, Theorem VI7, page 91] states, in effect, that if \( \dim M - \dim G = k > 0 \), then at least one element of \( G \) has dimension not less than \( k \). From Theorem IV, this result, and the fact that every atriodic continuous curve is one-dimensional, we obtain

**Theorem V.** If \( G \) is a one-dimensional continuum, then \( M \) is two-dimensional at each of its points.

\(^1\) The theorem of [1] shows emphatically that Theorem I cannot be strengthened to the extent of deleting the condition that the continuous curves of \( G \) be atriodic.

\(^2\) An \( n \)-dimensional Cantor manifold is a compact metric \( n \)-dimensional space which is not separated by any \((n-2)\)-dimensional subspace.

\(^3\) A rational curve is a compact metric continuum \( K \) such that each point of \( K \) is contained in arbitrarily small neighborhoods relative to \( K \) whose boundaries are countable. It is to be noted that a rational curve is not necessarily locally connected.

\(^4\) Eldon Dyer has recently obtained some general and interesting theorems about the dimension of \( G \) if \( M \) is \( n \)-dimensional and \( G \) is a continuous collection of arcs (or dendrons).
We shall give a proof of Theorem I in some detail. The proofs of Theorems II and III are similar to that of Theorem I. We shall simply indicate the arguments for these.

A simple chain in \( M \) is a finite collection \( x_1, x_2, \ldots, x_n \) of open sets such that \( x_i \cap x_j \) exists if and only if \( |i - j| \leq 1 \). The sets \( x_1, x_2, \ldots, x_n \) are called the links of the chain. We note that if \( s \) is an open interval of the arc \( t \) which lies in \( M \), there is an open subset \( U \) of \( M \) such that \( U \cdot t = s \) and \( \overline{U} \cdot i = \overline{s} \). If \( t \) is an arc and \( \epsilon \) is any positive number, then there exists a simple chain \( C \) covering \( t \) each of whose links is of diameter less than \( \epsilon \) and intersects \( t \) in a connected set.

A special case of a result of J. H. Roberts [3, Theorem 2] states that if \( E \) is a continuous collection of arcs filling up a compact metric space, there is a subcollection \( E' \) of \( E \) such that \( E' \) is dense in \( E \) and \( E \) is equicontinuous\(^6\) at each element of \( E' \). An argument suggested by that outlined in [3] yields a similar result in the case where \( E \) is a continuous collection of atriodic continuous curves. We do not give this argument in detail but henceforth let \( G' \) be a subcollection of \( G \) at each element of which \( G \) is equicontinuous.

**Proof of Theorem I.** Suppose, contrary to the statement of Theorem I that some closed, totally disconnected point set \( T \) separates \( M \) into sets \( D_1 \) and \( D_2 \). Then, since \( M \) is connected and \( G \) is continuous, some element \( g_1 \) of \( G \) intersects each of \( D_1 \) and \( D_2 \) and, since \( G' \) is dense in \( G \), some element \( g \) of \( G' \) intersects each of \( D_1 \) and \( D_2 \). Let \( h \), with endpoints \( a_1 \) and \( a_2 \) in \( D_1 \) and \( D_2 \) respectively, be an arc in \( G \). For each \( i \), \( i = 1, 2 \), let \( A_i \) be an open set containing \( a_i \) with \( \overline{A_i} \) contained in \( D_i \). Let \( C: c_1, \ldots, c_5 \) be a simple chain covering \( h \) such that (1) \( c_1 - c_1 \cdot c_2 \) contains \( A_1 \), (2) \( c_5 - c_4 \cdot c_5 \) contains \( A_2 \), (3) \( \tilde{c}_1 + \tilde{c}_2 \) is a subset of \( D_1 \), (4) \( \tilde{c}_4 + \tilde{c}_5 \) is a subset of \( D_2 \), (5) for \( i = 1, \ldots, 5 \), \( c_i \cdot h \) is connected, and (6) for \( i = 2, 3, 4 \), \( c_i \) and \( g - h \) are mutually exclusive. Clearly such a chain exists. Let \( U \) containing \( g \) be an open subcollection of \( G \) such that each element \( u \) of \( U \) contains an arc \( u_x \) with endpoints in \( A_1 \) and \( A_2 \) respectively such that (1) \( u_x \), covered by \( C \), (2) \( u - u_x \) does not intersect \( \tilde{c}_5 \), and (3) \( u_x \) does not contain two disjoint arcs each intersecting \( A_1 \) and \( c_3 \), \( A_2 \) and \( c_3 \), or \( c_2 \) and \( c_4 \). From the equicontinuity of \( G \) at \( g \) it follows that such a set \( U \) exists.

Let \( \bar{G} \) containing \( g \) be a nondegenerate closed connected subcollection of \( U \) and let \( \bar{H} \) containing \( h \) be a collection of arcs in one-to-one correspondence with \( \bar{G} \) such that each element of \( \bar{H} \) is contained in

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\(^6\) The collection \( G \) as above is said to be equicontinuous at the element \( g \) of \( G \) provided that for any \( \epsilon > 0 \) and any point \( p \) of \( g \), there exists a \( \delta > 0 \) such that if \( x \) and \( y \) are points of the same element \( g' \) of \( G \) and are each within \( \delta \) of \( p \), then there is an arc in \( g' \) containing \( x + y \) and of diameter less than \( \epsilon \).
the corresponding element of \( G \), is covered by \( C \), and has its endpoints in \( A_1 \) and \( A_2 \) respectively. Let \( \tilde{H} \) be topologized so as to be homeomorphic with \( G \) under the correspondence above.

Let \( g' \) be an element of \( G \) distinct from \( g \) and let \( h' \) be the corresponding element of \( \tilde{H} \). Let \( W \) be an open set in \( c_3 - (c_2 + c_4) \cdot c_3 \) such that \( \bar{W} - W \) does not intersect \( T \), \( W \) contains \( T \cdot h' \), and \( W \) does not intersect \( h' \). Let \( Z_1 \) and \( Z_2 \) be the subsets of \( \bar{W} - W \) in \( D_1 \) and \( D_2 \) respectively. Each of \( Z_1 \) and \( Z_2 \) is closed.

For each element \( k \) of \( \tilde{H} \), let \( N(k) \) be the collection of those components of \( k \cdot W \) having limit points in each of \( Z_1 \) and \( Z_2 \). Let \( n(k) \) be the number \((\text{mod } 2)\) of elements in \( N(k) \). Let \( \tilde{H}_0 \) be the collection of all elements \( k \) of \( \tilde{H} \) for which \( n(k) = 0 \). The collection \( \tilde{H}_0 \) contains \( h' \) and \( \tilde{H} - \tilde{H}_0 \) contains \( h \). We wish to show that each of \( \tilde{H}_0 \) and \( \tilde{H} - \tilde{H}_0 \) is open and hence that \( \tilde{G} \) is not connected—a contradiction.

Let \( \tilde{h} \) be any element of \( \tilde{H} \). There exists a simple chain \( C(\tilde{h}) \) covering \( h \) such that (1) each link of \( C(\tilde{h}) \) is a subset of a link of \( C \) and intersects \( \tilde{h} \) and (2) no link of \( C(\tilde{h}) \) intersects (a) each of \( D_1 \) and \( D_2 \) but not \( T \), (b) each of \( Z_1 \) and \( Z_2 \), or (c) each of \( \bar{W} \) and \( T - T \cdot W \).

Let \( X(\tilde{h}) \) be the collection of links of \( C(\tilde{h}) \) intersecting \( \bar{W} \). Let \( Y(\tilde{h}) \) be the collection of all maximal simple chains whose links are elements of \( X(\tilde{h}) \). If \( k \) is any element of \( \tilde{H} \) containing a subarc \( k' \) with endpoints in \( A_1 \) and \( A_2 \) and with \( k' \) covered by \( C(\tilde{h}) \), then \( n(k) = n(\tilde{h}) \). This follows from the fact that each element of \( N(k) \) is covered by exactly one element of \( Y(\tilde{h}) \) and the number of elements of \( N(k) \) in an element \( y \) of \( Y(\tilde{h}) \) is 1 or 0 \((\text{mod } 2)\) according as the end links of \( y \) do or do not intersect different sets of the two sets \( Z_1 \) and \( Z_2 \). Hence \( \tilde{H}_0 \) and \( \tilde{H} - \tilde{H}_0 \) are each open and Theorem I is proved.

**Indication of proof of Theorem II.** Suppose, contrary to Theorem II that there exists a connected open set \( D \) in \( M \) separated by a closed and totally disconnected set \( T \). Let \( (D_1, D_2) \) be a separation of \( D \) by \( T \). Then some element of \( G \) must contain an arc in \( D \) with endpoints in \( D_1 \) and \( D_2 \) respectively. Since \( G \) is a collection of arcs, some element of \( G' \) must also contain an arc in \( D \) with endpoints in \( D_1 \) and \( D_2 \) respectively and the argument is essentially reduced to that for Theorem I. We note that if the elements of \( G \) are not restricted to being arcs, then there exist simple examples without such an element of \( G' \) existing and with an open set separated by a point.

**Indication of proof of Theorem III.** Suppose \( G \) is a two-dimensional Cantor manifold and \( M \) is separated by a rational curve \( J \) into the two mutually separated sets \( D_1 \) and \( D_2 \). Clearly some element of \( G \) must intersect each of \( D_1 \) and \( D_2 \) for otherwise the set of those elements of \( G \) lying completely in \( J \) must exist and be closed.
and in $G$ must separate $G$. But by Theorem I this set is 0-dimen-
sional, a contradiction. Hence some element of $G$ intersects each of
$D_1$ and $D_2$, and thus some element of $G'$ contains an arc $h$ with end-
points in $D_1$ and $D_2$ respectively. But by an argument similar to that
used in the proof of Theorem I we can exhibit an open subcollection
$U$ of $G$ with $U$ containing $g$ and with $U - U$ countable, a contradic-
tion.

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A NOTE ON BASIC SETS OF HOMOGENEOUS
HARMONIC POLYNOMIALS

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For any set of non-negative integers $(b_j)$ such that $b_1 \leq 1$ and
$\sum_{j=1}^{k} b_j = n$, let

$$H_{b_1 \ldots b_k}^n = \sum (-1)^{\lceil a_1/2 \rceil} \frac{n!}{\prod_{j=1}^{k} a_j!} \frac{\prod_{j=2}^{k} \left( \frac{b_j - a_j}{2} \right)!}{\prod_{j=2}^{k} \left( \frac{b_j - a_j}{2} \right)!} \prod_{j=1}^{k} x_j^{a_j}$$

where the summation is extended over all $(a_j)$ such that,
(a) $a_j \equiv b_j \mod 2$, $j = 1, 2, \ldots, k$,
(b) $\sum_{j=1}^{k} a_j = n$,
(c) $a_j \leq b_j$, $j = 2, 3, \ldots, k$.

The polynomials (1) were shown by the authors to form a basic set
of homogeneous harmonic polynomials in $k$ variables [1].

It is easily seen that the following differential recursion formulas
hold for these polynomials:

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geneous polynomials in $k$ variables. II; received by the editors November 29, 1954.

1 Numbers in brackets refer to bibliography at the end of the paper.