ON SPACES FILLED UP BY CONTINUOUS COLLECTIONS
OF ATRIODIC CONTINUOUS CURVES

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Throughout this paper, \( G \) will denote a nondegenerate continuous collection of atriodic continuous curves (i.e. arcs or simple closed curves) filling up a compact metric continuum \( M \). As is well known, we may regard \( G \) itself as a compact metric continuum, with the elements of the collection \( G \) as the points of the space \( G \) and with \( G \) as a space, the image of \( M \) under an open continuous mapping whose inverse sets are the elements of the collection \( G \).

The principal results of this paper are the following theorems.

**Theorem I.** No closed, totally disconnected point set separates \( M \).

**Theorem II.** If each element of \( G \) is an arc, then no closed, totally disconnected point set separates any connected open subset of \( M \).

**Theorem III.** If \( G \) is a two-dimensional Cantor manifold, then \( M \) is not separated by any rational curve.

As an immediate corollary of Theorem I, we have

**Theorem IV.** At no point of \( M \) is the dimension of \( M \) less than 2.

A well known result cited in [2, Theorem VI17, page 91] states, in effect, that if \( \dim M - \dim G = k > 0 \), then at least one element of \( G \) has dimension not less than \( k \). From Theorem IV, this result, and the fact that every atriodic continuous curve is one-dimensional, we obtain

**Theorem V.** If \( G \) is a one-dimensional continuum, then \( M \) is two-dimensional at each of its points.

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1 The theorem of [1] shows emphatically that Theorem I cannot be strengthened to the extent of deleting the condition that the continuous curves of \( G \) be atriodic.

2 An \( n \)-dimensional Cantor manifold is a compact metric \( n \)-dimensional space which is not separated by any \((n - 2)\)-dimensional subspace.

3 A rational curve is a compact metric continuum \( K \) such that each point of \( K \) is contained in arbitrarily small neighborhoods relative to \( K \) whose boundaries are countable. It is to be noted that a rational curve is not necessarily locally connected.

4 Eldon Dyer has recently obtained some general and interesting theorems about the dimension of \( G \) if \( M \) is \( n \)-dimensional and \( G \) is a continuous collection of arcs (or dendrons).
We shall give a proof of Theorem I in some detail. The proofs of Theorems II and III are similar to that of Theorem I. We shall simply indicate the arguments for these.

A simple chain in $M$ is a finite collection $x_1, x_2, \cdots, x_n$ of open sets such that $x_i \times x_j$ exists if and only if $|i-j| \leq 1$. The sets $x_1, x_2, \cdots, x_n$ are called the links of the chain. We note that if $s$ is an open interval of the arc $t$ which lies in $M$, there is an open subset $U$ of $M$ such that $U \cdot t = s$ and $\overline{U} \cdot t = \overline{s}$. If $t$ is an arc and $\epsilon$ is any positive number, then there exists a simple chain $C$ covering $t$ each of whose links is of diameter less than $\epsilon$ and intersects $t$ in a connected set.

A special case of a result of J. H. Roberts [3, Theorem 2] states that if $E$ is a continuous collection of arcs filling up a compact metric space, there is a subcollection $E'$ of $E$ such that $E'$ is dense in $E$ and $E$ is equicontinuous at each element of $E'$. An argument suggested by that outlined in [3] yields a similar result in the case where $E$ is a continuous collection of atriodic continuous curves. We do not give this argument in detail but henceforth let $G'$ be a subcollection of $G$ at each element of which $G$ is equicontinuous.

Proof of Theorem I. Suppose, contrary to the statement of Theorem I that some closed, totally disconnected point set $T$ separates $M$ into sets $D_1$ and $D_2$. Then, since $M$ is connected and $G$ is continuous, some element $g_1$ of $G$ intersects each of $D_1$ and $D_2$ and, since $G'$ is dense in $G$, some element $g$ of $G'$ intersects each of $D_1$ and $D_2$. Let $h$, with endpoints $a_1$ and $a_2$ in $D_1$ and $D_2$ respectively, be an arc in $g$. For each $i$, $i = 1, 2$, let $A_i$ be an open set containing $a_i$ with $A_i$ contained in $D_i$. Let $C$: $c_1, \cdots, c_6$ be a simple chain covering $h$ such that (1) $c_1 - c_1 \times c_2$ contains $A_1$, (2) $c_5 - c_4 \times c_6$ contains $A_2$, (3) $c_1 + c_2$ is a subset of $D_1$, (4) $c_4 + c_5$ is a subset of $D_2$, (5) for $i = 1, \cdots, 5$, $c_i \cdot h$ is connected, and (6) for $i = 2, 3, 4$, $c_i$ and $g - h$ are mutually exclusive. Clearly such a chain exists. Let $U$ containing $g$ be an open subcollection of $G$ such that each element $u$ of $U$ contains an arc $u_x$ with endpoints in $A_1$ and $A_2$ respectively such that (1) $u_x$ is covered by $C$, (2) $u - u_x$ does not intersect $\overline{c}_5$, and (3) $u_x$ does not contain two disjoint arcs each intersecting $A_1$ and $c_3$, $A_2$ and $c_3$, or $c_2$ and $c_4$. From the equicontinuity of $G$ at $g$ it follows that such a set $U$ exists.

Let $\tilde{G}$ containing $g$ be a nondegenerate closed connected subcollection of $U$ and let $\tilde{H}$ containing $h$ be a collection of arcs in one-to-one correspondence with $G$ such that each element of $\tilde{H}$ is contained in $\tilde{G}$ provided that for any $\epsilon > 0$ and any point $p$ of $g$, there exists a $\delta > 0$ such that if $x$ and $y$ are points of the same element $g'$ of $G$ and are each within $\delta$ of $p$, then there is an arc in $g'$ containing $x+y$ and of diameter less than $\epsilon$. 

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\textsuperscript{5} The collection $G$ as above is said to be equicontinuous at the element $g$ of $G$ provided that for any $\epsilon > 0$ and any point $p$ of $g$, there exists a $\delta > 0$ such that if $x$ and $y$ are points of the same element $g'$ of $G$ and are each within $\delta$ of $p$, then there is an arc in $g'$ containing $x+y$ and of diameter less than $\epsilon$. 

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the corresponding element of $\mathcal{C}$, is covered by $C$, and has its endpoints in $A_1$ and $A_2$ respectively. Let $\mathcal{H}$ be topologized so as to be homeomorphic with $\mathcal{C}$ under the correspondence above.

Let $g'$ be an element of $\mathcal{C}$ distinct from $g$ and let $h'$ be the corresponding element of $\mathcal{H}$. Let $W$ be an open set in $c_3 - (c_2 + c_4) \cdot c_3$ such that $\mathcal{W} - W$ does not intersect $T$, $W$ contains $T \cdot h$, and $W$ does not intersect $h'$. Let $Z_1$ and $Z_2$ be the subsets of $\mathcal{W} - W$ in $D_1$ and $D_2$ respectively. Each of $Z_1$ and $Z_2$ is closed.

For each element $k$ of $\mathcal{H}$, let $N(k)$ be the collection of those components of $k \cdot W$ having limit points in each of $Z_1$ and $Z_2$. Let $n(k)$ be the number (mod 2) of elements in $N(k)$. Let $\mathcal{H}_0$ be the collection of all elements $k$ of $\mathcal{H}$ for which $n(k) = 0$. The collection $\mathcal{H}_0$ contains $h'$ and $\mathcal{H} - \mathcal{H}_0$ contains $h$. We wish to show that each of $\mathcal{H}_0$ and $\mathcal{H} - \mathcal{H}_0$ is open and hence that $\mathcal{C}$ is not connected—a contradiction.

Let $h$ be any element of $\mathcal{H}$. There exists a simple chain $C(h)$ covering $h$ such that (1) each link of $C(h)$ is a subset of a link of $C$ and intersects $h$ and (2) no link of $C(h)$ intersects (a) each of $D_1$ and $D_2$ but not $T$, (b) each of $Z_1$ and $Z_2$, or (c) each of $\mathcal{W}$ and $T - T \cdot W$.

Let $X(h)$ be the collection of links of $C(h)$ intersecting $\mathcal{W}$. Let $Y(h)$ be the collection of all maximal simple chains whose links are elements of $X(h)$. If $k$ is any element of $\mathcal{H}$ containing a subarc $k'$ with endpoints in $A_1$ and $A_2$ and with $k'$ covered by $C(h)$, then $n(k) = n(h)$. This follows from the fact that each element of $N(k)$ is covered by exactly one element of $Y(h)$ and the number of elements of $N(k)$ in an element $y$ of $Y(h)$ is 1 or 0 (mod 2) according as the end links of $y$ do or do not intersect different sets of the two sets $Z_1$ and $Z_2$. Hence $\mathcal{H}_0$ and $\mathcal{H} - \mathcal{H}_0$ are each open and Theorem I is proved.

**Indication of proof of Theorem II.** Suppose, contrary to Theorem II that there exists a connected open set $D$ in $M$ separated by a closed and totally disconnected set $T$. Let $(D_1, D_2)$ be a separation of $D$ by $T$. Then some element of $G$ must contain an arc in $D$ with endpoints in $D_1$ and $D_2$ respectively. Since $G$ is a collection of arcs, some element of $G'$ must also contain an arc in $D$ with endpoints in $D_1$ and $D_2$ respectively and the argument is essentially reduced to that for Theorem I. We note that if the elements of $G$ are not restricted to being arcs, then there exist simple examples without such an element of $G'$ existing and with an open set separated by a point.

**Indication of proof of Theorem III.** Suppose $G$ is a two-dimensional Cantor manifold and $M$ is separated by a rational curve $J$ into the two mutually separated sets $D_1$ and $D_2$. Clearly some element of $G$ must intersect each of $D_1$ and $D_2$ for otherwise the set of those elements of $G$ lying completely in $J$ must exist and be closed.
and in $G$ must separate $G$. But by Theorem I this set is 0-dimen-
sional, a contradiction. Hence some element of $G$ intersects each of
$D_1$ and $D_2$, and thus some element of $G'$ contains an arc $h$ with end-
points in $D_1$ and $D_2$ respectively. But by an argument similar to that
used in the proof of Theorem I we can exhibit an open subcollection
$U$ of $G$ with $U$ containing $g$ and with $\overline{U} - U$ countable, a contradic-
tion.

References

2. Witold Hurewicz and Henry Wallman, Dimension theory, Princeton University
   Press, 1948.

A NOTE ON BASIC SETS OF HOMOGENEOUS
HARMONIC POLYNOMIALS

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For any set of non-negative integers $(b_j)$ such that $b_1 \leq 1$ and
$\sum_{j=1}^{k} b_j = n$, let

\begin{equation}
H^n_{b_1 \ldots b_k} = \sum (-1)^{a_j/2} \frac{n!}{\prod_{j=1}^{k} a_j!} \frac{\left[ \frac{a_1}{2} \right]!}{\prod_{j=2}^{k} \left( \frac{b_j - a_j}{2} \right)!} \prod_{j=1}^{k} x_j^{a_j}
\end{equation}

where the summation is extended over all $(a_j)$ such that,

(a) $a_j \equiv b_j \mod 2$, $j = 1, 2, \ldots, k$,
(b) $\sum_{j=1}^{k} a_j = n$,
(c) $a_j \leq b_j$, $j = 2, 3, \ldots, k$.

The polynomials (1) were shown by the authors to form a basic set
of homogeneous harmonic polynomials in $k$ variables [1].

It is easily seen that the following differential recursion formulas
hold for these polynomials:

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genous polynomials in $k$ variables. II; received by the editors November 29, 1954.

1 Numbers in brackets refer to bibliography at the end of the paper.