ON SPECIAL $W$-SURFACES

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A $W$-surface is a surface in ordinary Euclidean space for which there is a functional relation between the principal curvatures $k_1, k_2$:

$$W(k_1, k_2) = 0.$$  

We shall be interested in those $W$-surfaces for which (1) can be written in the form

$$f(H, \mu) = 0, \quad \mu = H^2 - K,$$

where $H$ and $K$ are the mean curvature and the Gaussian curvature respectively and where $f$ is of class $C^1$. For such $W$-surfaces we have

$$\frac{\partial W}{\partial k_1} = \frac{f_H}{2} + (k_1 - k_2)f_{\mu}/2,$$

$$\frac{\partial W}{\partial k_2} = \frac{f_H}{2} - (k_1 - k_2)f_{\mu}/2.$$  

It follows that at an umbilic ($k_1 = k_2$) we have

$$\frac{\partial W}{\partial k_1} \cdot \frac{\partial W}{\partial k_2} = \frac{f_H^2}{4} \geq 0.$$  

A $W$-surface (2) is called special if $f_H \neq 0$ at every umbilic.

Apparently there are very few closed special $W$-surfaces. The following theorem is due to H. Hopf:

**The only closed special analytic $W$-surfaces of genus zero are spheres.**

It was P. Hartman and A. Wintner who succeeded in removing the analyticity assumption in Hopf’s Theorem. Their theorem can be stated as follows:

**Let $S$ be a closed special $W$-surface of genus zero, which is $C^3$-imbedded in Euclidean space. Then $S$ is a sphere.**

We shall show that the formalism developed in the preceding paper gives a very simple proof of the theorem of Hartman-Wintner.

Received by the editors November 15, 1954.

1 The notion of a special $W$-surface was first introduced by the author in the paper *Some new characterizations of the Euclidean sphere*, Duke Math. J. vol. 12 (1945) pp. 279–290. I take this opportunity to mention that the proof of Theorem 3 in that paper is not valid. So far as I know, the question whether there exist closed surfaces of constant mean curvature and of genus $>0$ remains unanswered.


4 The hypotheses of Hartman-Wintner are weakened in one respect in that the function $f$ in (2) is here assumed to be of class $C^1$, while they supposed $f$ to be of class $C^2$.  

783
Let \( x, y \) be isothermal parameters, and let \( z = x + iy \). Let \( E(dx^2 + dy^2), L dx^2 + 2M dx dy + N dy^2 \) be respectively the first and second fundamental forms of the surface, so that

\[
2EH = L + N, \\
E^2K = LN - M^2.
\]

Put

\[
w = (L - N)/2 - iM.
\]

Then Codazzi’s equations can be written

\[
w_z = EH_z.
\]

Also we have from (5)

\[
\mu = H^2 - K = \frac{w\bar{w}}{E^2}.
\]

If the surface is a special \( W \)-surface satisfying (2), we have, in a neighborhood of an umbilic,

\[
H_z = -\left(\frac{f_u}{f_H}\right)\mu_z.
\]

This means that \( w \) satisfies a nonlinear differential equation of the form

\[
w_z = P(w\bar{w})_z + Qw\bar{w},
\]

where

\[
P = -\frac{f_u}{E f_H}, \quad Q = \frac{2E_z}{E^2 f_u f_H}.
\]

Following the procedure of Hartman-Wintner, the proof of their theorem depends on the lemmas:

**Lemma 1.** Let \( w(z, \bar{z}) \) be a solution of (9), in a sufficiently small neighborhood of \( z = 0 \), at which \( w = 0 \). Then \( \lim_{z \to 0} w(z, \bar{z})z^{-k} \) exists if \( w = o\left(|z|^{k-1}\right) \).

**Lemma 2.** Under the hypotheses of Lemma 1, suppose that \( w = o\left(|z|^{k-1}\right) \) for all \( k \). Then \( w(z, \bar{z}) = 0 \) in a neighborhood of \( z = 0 \).

From these lemmas we derive immediately the theorem of Hartman-Wintner. In fact, it follows from Lemma 2 that if 0 (\( z = 0 \)) does not have a neighborhood which consists entirely of umbilics, there exists an integer \( k \), such that \( w = o\left(|z|^{k-1}\right), w \neq o\left(|z|^k\right) \). By Lemma 1, \( \lim_{z \to 0} w(z, \bar{z})z^{-k} \) exists and is \( \neq 0 \). We can therefore write

\[
w(z, \bar{z}) = cz^k + o\left(|z|^k\right) \quad c \neq 0.
\]
It follows that the umbilic 0 is isolated and has an index \(-k<0\). By well-known arguments this implies the theorem of Hartman-Wintner.

It remains to prove the above lemmas. For this purpose let \( D \) be a disc of radius \( R \) about 0, and \( C \) its boundary circle. There exists a constant \( A > 0 \), such that in \( D \),

\[
| P(w\bar{w})z + Qw\bar{w} | \leq A | w | .
\]

Suppose that \( w = o(|z|^{k-1}) \). Let \( \zeta = \xi + i\eta \) be an interior point of \( D \). Then we have, for \( \zeta \neq 0 \),

\[
\frac{1}{z^k(z - \zeta)} \frac{wdz}{dz} = \frac{P(w\bar{w})z + Qw\bar{w}}{z^k(z - \zeta)} d\bar{z} \wedge dz.
\]

Application of Stokes Theorem gives

\[
-2\pi iw(\zeta, \bar{\zeta})z^{-k} + \int_C \frac{wdz}{z^k(z - \zeta)} = \int_D \int_D \frac{P(w\bar{w})z + Qw\bar{w}}{z^k(z - \zeta)} d\bar{z}dz.
\]

It follows that

\[
2\pi | w(\zeta, \bar{\zeta})z^{-k} | \leq \int_C \left| \frac{w(z, \bar{z})}{z^k(z - \zeta)} \right| dz
\]

\[
+ 2A \int_D \int_D \left| \frac{w(z, \bar{z})}{z^k(z - \zeta)} \right| dx dy.
\]

We multiply this inequality by \( d\xi d\eta / |\zeta - z_0| \), \( z_0 \in D \), and integrate over \( D \). Remembering that

\[
\int_D \int_D \frac{dx dy}{|z - \zeta|} < 2R,
\]

\[
\frac{1}{|z - \zeta|} \frac{1}{|z - z_0|} = \left| \frac{1}{z-z_0} + \frac{1}{\zeta - z_0} \right|,
\]

we get from this integration

\[
2\pi \int_D \int_D \left| \frac{w(z, \bar{z})}{z^k(z - \zeta)} \right| dx dy \leq 4R \int_C \left| \frac{w(z, \bar{z})}{z^k(z - \zeta)} \right| dz
\]

\[
+ 8AR \int_D \int_D \left| \frac{w(z, \bar{z})}{z^k(z - \zeta)} \right| dx dy
\]

or
We choose $R$ so small that $2\pi - 8AR > 0$. Then
\[ \int \int_D |w(z, \bar{z})/z^k(z - \zeta)| \, dx \, dy \]
is bounded as $\zeta \to 0$, and the same is true of $|w(\zeta, \bar{\zeta})\zeta^{-k}|$. It follows that $|(w\bar{w})_1\zeta^{-k}|$ is bounded and from (14) that $\lim_{r \to 0} w(\zeta, \bar{\zeta})\zeta^{-k}$ exists. This proves Lemma 1.

To prove Lemma 2 we multiply (15) by $d\xi d\eta$ and integrate over $D$. This gives
\[ 2\pi \int \int_D |w(z, \bar{z})z^{-k}| \, dx \, dy \leq 2R \int_C |w(z, \bar{z})z^{-k}| \, |dz| \]
\[ + 4AR \int \int_D |wz^{-k}| \, dx \, dy \]
or
\[ (2\pi - 4AR) \int \int_D |w(z, \bar{z})z^{-k}| \, dx \, dy \leq 2R \int_C |w(z, \bar{z})z^{-k}| \, |dz| . \]

Suppose there exists a $z_0$ such that $w(z_0, \bar{z}_0) \neq 0$, $|z_0| < R$. Then the left-hand side of the above inequality is $\geq a |z_0|^{-k}$, and the right-hand side is $\leq bR^{-k}$, where $a$ and $b$ are positive constants independent of $k$. The hypothesis of Lemma 2 implies that $|z_0/R|^k \geq a/b$ for all $k$, which is a contradiction. It follows that $w(z, \bar{z})$ vanishes identically for $|z| < R$.

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