GENERATORS OF THE RING OF BOUNDED OPERATORS

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J. A. Dieudonné suggested in conversation that some small number of projections might suffice to generate the ring $\mathcal{B}$ of all bounded operator on *separable* Hilbert space $\mathcal{H}$. There is some analogy between such a result and the theorem that a compact connected metric group can be generated by two elements.¹ The analogy is still closer for Theorem 2 below.

In this paper, operators $A, \cdots$ will be said to "generate" the smallest ring (i.e., *weakly* closed self-adjoint algebra) containing $A, \cdots$ and the constants.

**Theorem 1.** There exist three projections which generate the ring $\mathcal{B}$. The number three cannot be reduced if $\mathcal{H}$ has dimensionality 3 or greater.

If $\dim (\mathcal{H}) = 1$, there is nothing but constants in $\mathcal{B}$. If $\dim (\mathcal{H}) = 2$, then any two noncommuting projections generate $\mathcal{B}$. Hereafter suppose $\dim (\mathcal{H}) \geq 3$.

I will use what I shall call the *closeness operator* $C = C(E, F)$ associated with any two projections $E, F$. It is defined by $C(E, F) = 1 - E - F + EF + FE = EFE + (1 - E)(1 - F)(1 - E)$. It is a positive definite operator which, in case² $E \cap F + EF \cap (1 - F) + (1 - E) \cap F + (1 - E) \cap (1 - F) = 0$, acts like "the square of the cosine of the angle" between $E\mathcal{H}$ and $F\mathcal{H}$.

To show $E$ and $F$ fail to generate $\mathcal{B}$, I shall show some nonconstant operator commutes with both; this is enough because such an operator commutes with the whole ring generated by $E$ and $F$, whereas the commutator of $\mathcal{B}$ contains only constants. Since $C(E, F)$ commutes with $E$ and $F$, the only case to be considered is $C$ constant. $E \neq 0$ may be assumed. Choose $x = Ex$, $x \neq 0$, $x$ otherwise arbitrary. Now the subspace spanned by $x$ and $Fx$ is not zero, and since it is at most 2-dimensional it is not $\mathcal{H}$; so the projection $P$ on it is nonconstant. $P\mathcal{H}$ is invariant under $E$ and $F$, since $EFx = EFEx = Cx$, which in this case is a multiple of $x$. Therefore $P$ is a nonconstant operator


² Here "$\cap$" is intersection in the lattice of projections. This equation says $E\mathcal{H}$ and $F\mathcal{H}$ are in position $p$ (J. Dixmier, *Position relative de deux variétés linéaires fermées dans un espace de Hilbert*, Rev. Sci. vol. 86 (1948) pp. 387-399).
commuting with $E$ and $F$.

Now for the proof of the first sentence in the theorem. Only countable dimensionality will be treated, finite dimensionality is handled similarly. There is a good deal of leeway in the construction; the particular generators $E_1$, $E_2$, $E_3$ given here are chosen for convenience.

Let $x_1$, $y_1$, $x_2$, $y_2$, $x_3$, $\cdots$ be an orthonormal basis of $\mathcal{H}$. Let $z_n = \cos \theta_n x_n + \sin \theta_n y_n$, $n = 1, 2, \cdots$, with $\theta_n = \pi/(2n+1)$. Let $P_n$ be the projection on $[x_n, y_n]$, the subspace spanned by $x_n$ and $y_n$, $n = 1, 2, \cdots$. Let $E_1$ be the projection on $[x_1, x_2, x_3, \cdots]$; $E_2$, the projection on $[z_1, z_2, z_3, \cdots]$.

Now the ring generated by $E_1$ and $E_2$ contains $\mathbb{C} = C(E_1, E_2)$. It can be shown by a direct computation that the eigenspaces of $C$ are the $P_n \mathcal{H}$, the corresponding eigenvalues being $\cos^2 \theta_n$. Each eigenvalue is an isolated point of the spectrum; the characteristic function of the set containing only $\cos^2 \theta_n$ is measurable (even continuous) on the spectrum of $C$. The spectral theorem implies that the ring contains all the $P_n$. Also the ring contains every operator on $P_n \mathcal{H}$, for on that 2-dimensional Hilbert space $E_1$ and $E_2$ are noncommuting projections (see the remark at the beginning of the proof).

Finally, define $E_3$ as the projection on

$$[x_1 + x_2, y_2 + y_3, \cdots, x_{2n-1} + x_{2n}, y_{2n} + y_{2n+1}, \cdots].$$

The ring $\mathcal{R}$ generated by $E_1$, $E_2$, and $E_3$ will be shown to be $\mathcal{B}$.

Let $E(x; y)$ denote, for any unit vectors $x$ and $y$, the operator characterized by

$$E(x; y)x = y,$$

$z \perp x$ implies $E(x; y)z = 0$.

Also hereafter let $w_n$ mean either $x_n$ or $y_n$, $n = 1, 2, \cdots$.

$\mathcal{R}$ contains $E(x_{2n-1}; x_{2n})$. For it contains the projection on $[x_{2n-1}]$, and premultiplying that projection by $2P_{2n}E_3$ gives the desired operator. Therefore $\mathcal{R}$ contains $E(w_{2n-1}; w_{2n})$, and necessarily also its adjoint, $E(w_{2n}; w_{2n-1})$. Similarly, $\mathcal{R}$ contains $E(y_{2n}; y_{2n+1})$, hence $E(w_{2n}; w_{2n+1})$ and $E(w_{2n+1}; w_{2n})$. By induction, $E(w_i; w_j) \in \mathcal{R}$. Therefore $\mathcal{R}$ contains every operator whose matrix, using the originally given orthonormal basis of $\mathcal{H}$, has finitely many nonzero entries.

Let $A \in \mathcal{B}$, and let

$$A_n = \sum_{k=1}^{n} P_k A P_k.$$
Then $A_n \in \mathcal{R}$, and $A$ is the weak limit of the $A_n$, so $A \in \mathcal{R}$. It has been proved that $\mathcal{R} = \mathcal{B}$.

(Weak closure of $\mathcal{R}$ was required only in the last paragraph of the proof; uniform closure was all that was used before. I do not know if there exist three projections which generate $\mathcal{B}$ by algebraic operations and uniform limits.*)

**Theorem 2.** There exist two unitary operators which generate $\mathcal{B}$. They may be chosen so one of them is a symmetry.

This will be proved relying largely on the previous proof, and keeping the same symbols. Again I shall treat only the countable case.

Let $Ux_n = z_n$, $Uy_n = -\sin \theta_n x_n + \cos \theta_n y_n$. This defines $U$ as a unitary operator. Let $V = 1 - 2E_3$, a symmetry. The ring $\mathcal{R}'$ generated by $U$ and $V$ will be shown to be $\mathcal{B}$.

$E_3 = (1 - V)/2 \in \mathcal{R}'$. As above, one shows first that each $P_n \mathcal{K}$ is an eigenspace of $U + U^*$, corresponding to the eigenvalue $2 \cos \theta_n$; and therefore that $P_n \in \mathcal{R}'$. The last part of the previous proof can be invoked once it is shown that $\mathcal{R}'$ contains every operator on the 2-dimensional subspace $P_n \mathcal{K}$. But $2P_nE_3P_n \in \mathcal{R}'$ and $2UP_nE_3P_nU^* \in \mathcal{R}'$ are noncommuting projections operating on $P_n \mathcal{K}$.

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* The projections given here do not. In fact, every operator $A$ which is a uniform limit of polynomials in $E_1$, $E_2$, and $E_3$, has the special property (among others) that $(Ax_{2n}, x_{2n})$ has a limit as $n \to \infty$. Proof omitted.