ON THE MAXIMAL DILATION OF QUASICONFORMAL MAPPINGS

KURT STEBEL

1. Let $G$ and $G'$ be two plane open sets and $w(z)$ a topological mapping of $G$ onto $G'$. By $Q$ we denote any quadrilateral in $G$, i.e. the topological image of a closed square with a distinguished pair of opposite sides. The conformal modulus $m$ of $Q$ is the ratio $m = a/b$ of the sides of a conformally equivalent rectangle $R$, the distinguished sides of $Q$ corresponding to the sides of length $b$. We call this essentially unique conformal mapping of $Q$ onto $R$ the canonical mapping of $Q$. The modulus $m$ is equal to the extremal distance of the two distinguished sides of $Q$ with respect to $Q$. The maximal dilation of the mapping $w(z)$ on $G$ is the number

$$K[w(z)] = K = \sup_{Q} \frac{m'}{m}$$

where $m'$ denotes the modulus of the image $Q'$ of $Q$ under the mapping $w(z)$ and $Q$ varies over all possible quadrilaterals. The mapping is said to be quasiconformal if $K$ is finite.

Given a closed subset $E \subset G$ (closed only with respect to $G$). Then, $G - E$ is open, and if we denote by $K_0[w(z)] = K_0$ the maximal dilation of $w(z)$ on $G - E$, we get $K_0 \leq K$. We are looking for sufficient conditions on $E$ such that $K_0 = K$. The answer will be different if we consider only the class of all quasi-conformal mappings of $G$ or the larger class of all topological mappings, including the ones with infinite maximal dilation. We call a point set $E$ which allows the conclusion $K_0 = K$ deletable for the class in consideration. It was proved by Ahlfors in [1] that analytic arcs are deletable for the class of all topological mappings. It is also proved there that $K = \sup_{\bar{Q}} \frac{m'}{m}$, where $\bar{Q}$ denotes any analytic quadrilateral, i.e. a quadrilateral $Q$ with a canonical mapping which is conformal in an open neighborhood of $Q$, and furthermore that $K$ does not become larger if the boundary curves of the quadrilaterals are allowed to have points in common with the boundary of $G$.

2. If $E$ is a discrete point set, we can consider the slightly more general problem that $w_0(z)$ is only known to be quasiconformal with maximal dilation $K_0$ on $G - E$, without knowing that $w_0(z)$ is a topological mapping of the whole open set $G$. What are the conditions on

---

Presented to the Society, February 26, 1955; received by the editors November 8, 1954 and, in revised form, December 31, 1954.

1 Theorem 4, p. 9.
$E$ in order that for every $w_0(z)$ there is a topological mapping $w(z)$ of $G$ which coincides with $w_0(z)$ on $G-E$ and has the same maximal dilation. We call $w(z)$ the continuation of $w_0(z)$. It is clear that there can exist only one topological continuation.

Theorem 1. A necessary and sufficient condition that every quasi-conformal mapping $w_0(z)$ of $G-E$ has a continuation to $G$ with the same maximal dilation is that every compact subset of $E$ is a nullset $O_{AD}$.²

The condition is necessary, for, if $E_0$ is any compact subset of $E$ which is not a set $O_{AD}$, there exists a parallel slit mapping of the complement of $E_0$, which is conformal outside $E_0$ and cannot be extended conformally over $E_0$.

If, on the other hand, $E$ possesses the property of the theorem, then to every point $z_0 \subset E$ there exists a sequence of nonoverlapping doubly connected ring-domains in $G-E$ with a divergent sum of moduli $\mu_i(\mu_i=$ extremal distance of the two boundary components of a ring domain), a property which is invariant under the quasiconformal mapping of $G-E$. But a boundary component with this property must necessarily be a point. Therefore every point of $E$ goes over into a point, and it is readily seen that $w_0(z)$ has a topological continuation $w(z)$ over $E$. It follows from the succeeding lemma that the maximal dilation of $w(z)$ on $G$ is $K_0$.

Lemma 1. If $Q$ is an analytic quadrilateral in $G$ and $E_0$ a compact $O_{AD}$ set in $Q$, then to every $\epsilon>0$ there exists a finite system of simply connected, nonoverlapping longitudinal³ strips $S_i, S_i \subset Q-E_0$ with modulus $m_i$, such that $\sum_i 1/m_i \geq 1/m - \epsilon$.

From that we get, because of $m'_i \leq K_0 m_i$,

$$\frac{1}{m'} \geq \sum_i \frac{1}{m'_i} \geq K_0^{-1} \sum_i \frac{1}{m_i} \geq K_0^{-1} \left( \frac{1}{m} - \epsilon \right)$$

and therefore

$$m' \leq K_0 m$$

for each analytic quadrilateral in $G$.

The lemma can be proved in the same way as Theorem 9 in Ahlfors and Beurling [2]. We map $Q$ onto a rectangle $R$ with sides $a$ and $b$ by means of its canonical conformal mapping. $E_0$ is transformed

² I.e. a set which allows no nonconstant and single-valued analytic function with a bounded Dirichlet-integral in its complement.

³ That is to say the boundary of $S_i$ has an interval in common with each distinguished side of $Q$; $S_i$ is therefore a quadrilateral with these intervals as distinguished sides.
into a set $O_{AD}$ which we denote by $E_0'$. To any given $\epsilon > 0$ we can find a concentric rectangle $R'$ with sides $a' > a$ and $b' < b$ and such that $b'/a' \geq b/a - \epsilon/2$. A curvilinear rectangle $R''$ which is contained in the rectangle with sides $a'$, $b'$ and contains the rectangle with sides $a$, $b$ and the sides of which do not meet $E_0$ can be constructed; we choose its distinguished sides outside $R$ and such that its modulus is $\leq a'/b'$. The set $E_0' \cap R''$ is a compact subset of the open rectangle $R''$, and by an exhaustion we can obviously find the strips $S_i''$ in $R''$ with $\sum_i 1/m_i'' \geq 1/m'' - \epsilon/2$. Each $S_i''$ contains a longitudinal strip $S_i$ of the original rectangle $R$ with modulus $m_i \leq m_i''$. Therefore we get

$$\sum_i \frac{1}{m_i} \geq \sum_i \frac{1}{m_i''} \geq \frac{1}{m''} - \epsilon/2 \geq b'/a' - \epsilon/2 \geq b/a - \epsilon = \frac{1}{m} - \epsilon.$$

3. For the composition of piecewise quasiconformal mappings however the stress lies on connected, not on discrete point sets. To get an answer in this direction, we consider a rectangle $R$ in the $z$-plane ($0 \leq x \leq a$, $0 \leq y \leq b$) and a topological mapping $w(z)$ of $R$ onto a rectangle $R'$ in the $w$-plane ($0 \leq u \leq a'$, $0 \leq v \leq b'$) which preserves the four sides respectively. By $E_y$ we denote the intersection of the line $\Re z = y$ with the given closed set $E$ in $R$, and by $L_u(y)$ the linear measure of the vertical projection (i.e. onto the $u$-axis) of the $w$-image $E_y'$ of $E_y$. The modulus of a horizontal rectangle is its length divided by its height; the modulus of its $w$-image has to be taken with respect to the sides which correspond to the vertical sides of the rectangle.

**Lemma 2.** If (1) the modulus $m$ of every horizontal rectangle in $R$ which has no interior point in common with $E$ and the modulus $m'$ of its image satisfy $m' \leq Km$, where $K$ is some positive constant, and

(2) the linear measure $L_u(y)$ of the vertical projection of $E_y'$ is zero for almost every $y$, we have

$$a'/b' \leq Ka/b.$$

**Proof.** For any $\epsilon > 0$ the set $O_\epsilon$ of all values $y$ with $L_u(y) < \epsilon$ is open in $0 \leq y \leq b$ and has linear measure $b$. For any $y \in O_\epsilon$ we can find an interval $y_1 < y < y_2$ and a family of finitely many rectangles $R_i$ with sides on $\Re z = y_1$ and $\Re z = y_2$ which contain every $E_y$ for $y_1 < y < y_2$, and such that their $w$-images have a vertical projection of total linear measure less than $\epsilon$. Let now $B$ be a closed subset of $O_\epsilon$ of measure $> b - \epsilon$. We have an open covering of $B$ by means of intervals $\beta$ of the above kind, and by the Heine-Borel theorem there exists a finite covering, say $\beta_1, \ldots, \beta_n$. Starting with $\beta_1$ and the rectangles in the corresponding horizontal strip ($0 \leq x \leq a$, $y \in \beta_1$), we take in
every new interval $\beta_i$ only that part which does not lie in one of the former intervals, and restrict the corresponding horizontal strip and its rectangles in the same way. We get a system of finitely many strips $S_i$ of height $b_i$ with total vertical measure $\sum_i b_i \geq b - \epsilon$. The rectangles $T_{ij}$ in $S_i$, complementary to the rectangles $R_{ij}$ in $S_i$, have no interior point in common with $E$ and thus their moduli satisfy $m_{ij}' \leq K m_{ij}$. As the vertical projection of the images $R_{ij}'$ of the $R_{ij}$ has a total linear measure $< \epsilon$, there exist finitely many closed, disjoint intervals on the $u$-axis of total measure $< \epsilon$ covering this projection. We denote the complementary intervals on the $u$-axis by $\alpha_i'$, their length by $a_i'$: we have $\sum_i a_i' > a' - \epsilon$. Each $\alpha_i'$ is spanned by the image $T_{ij}'$ of at least one $T_{ij}$, and the modulus of this $T_{ij}'$ is therefore $\geq a_{ij}'^2 / A_{ij}'$, $A_{ij}'$ denoting the area of (the interior of) $T_{ij}'$. From that we get for each strip $S_i$

$$\frac{a_i}{b_i} \geq \frac{1}{K} \sum_j m_{ij} \geq \frac{1}{K} \sum_j m_{ij}' \geq \frac{1}{K} \sum_j \frac{a_{ij}'}{A_{ij}'}$$

$$\geq \frac{1}{K} \left( \sum_j a_{ij}' \right)^2 / \sum_j A_{ij}' \geq \frac{1}{K} \frac{(a' - \epsilon)^2}{A_i'}$$

with $A_i' =$ area of $S_i'$. Taking the reciprocals and summing up we get

$$\frac{b - \epsilon}{a} \leq \frac{\sum_i b_i}{a} \leq \frac{\sum_i A_i'}{(a' - \epsilon)^2} \leq K \frac{a'b'}{(a' - \epsilon)^2}$$

and therefore

$$b/a \leq K(b'/a')$$

which proves the lemma.

Let $E$ be an arbitrary set of finite linear measure $L$ in $R$. Then it is readily proved that the set of values $y$ where $E_y$ consists of at least $N$ points has linear measure $\leq L/N$, and therefore the set of $y$ where $E_y$ consists of infinitely many points has measure zero. If $E$ is of $\Sigma$-finite measure, i.e. the sum of denumerably many sets $E^i$ of finite linear measure, the set of values $y$ for which $E_y$ consists of nonde- numerably many points has linear measure zero. But for any $y$ where $E_y$ is denumerable, the image set is denumerable and so is its projection, therefore $L_u(y) = 0$. As the property to be of $\Sigma$-finite linear measure is carried over by a conformal mapping of the closed quadrilateral, we get the

**Theorem 2.** If $E$ is a closed subset of $G$ of $\Sigma$-finite linear measure and $w(z)$ any topological mapping of $G$, its maximal dilation on $G$ is equal
to its maximal dilation on $G - E$, that is to say $E$ is deletable for the class of all topological mappings of $G$.

If $E = E_1 + E_2$ is the sum of a closed set $E_1$ of $\Sigma$-finite linear measure and a closed set $E_2$, every compact subset of which is a nullset $O_{AD}$, then it is clear from the construction that $E$ is deletable. We can first delete $E_1 - E_2$, which is a closed set of $G - E_2$, and then $E_2$.

From Theorem 2 we get the following generalization of Theorem 13 in Ahlfors and Beurling [2]:

**Corollary.** A closed, discrete point set $E$ of the complex plane, which is of $\Sigma$-finite linear measure, is a nullset $O_{SB}$ if and only if it is a nullset $O_{AD}$.

For, if it is a nullset $O_{SB}$ every schlicht conformal mapping of the complement of $E$ is continuous on $E$ and has therefore a conformal continuation, i.e. is a linear transformation. But this is known to be a sufficient condition for a closed discrete pointset to be an $O_{AD}$ set. The converse is obvious.

4. If the mapping $w(z)$ is not only known to be topological in $G$ but quasiconformal, that means has finite maximal dilation, and if outside $E$ the maximal dilation is $K_0$, the point sets $E$ which allow us to conclude $K_0 = K$ are much larger.

**Lemma 3.** Let $w(z)$ be a topological mapping of $R$ onto $R'$ as in Lemma 2. If the two following conditions are fulfilled:

1. For every horizontal rectangle in $R$ and a certain positive number $K$ we have $m' \leq K m$;
2. $E$ is of two-dimensional measure zero; then the linear measure $L(y)$ of $E'_y$ is zero for almost every $y$.

**Proof.** If this were not the case, we could find a closed subset $B$ of $0 \leq y \leq b$ of positive linear measure $h$ and a positive number $l$ such that the linear measure of $E_y$ would be zero while the linear measure of $E'_y$ would be $L(y) \geq l$ for every $y \in B$. This is so because $L(y)$ is a measurable function of $y$ and the linear measure of $E_y$ is zero for almost all $y$.

If $y \in B$ is arbitrary, we can cover $E_y$ by finitely many open (relatively to $0 \leq x \leq a$) intervals $\alpha_i$ of length $a_i$. The images have length $a'_i$. We then take an interval $y_1 < y < y_2$ and consider the rectangles $R_j(x \in \alpha, y_1 \leq y \leq y_2)$. We call them the rectangles corresponding to the intervals $\alpha_i$ in the horizontal strip $(0 \leq x \leq a, y_1 \leq y \leq y_2)$. Every $R_j$ is mapped onto a quadrilateral $R'_j$ and we denote by $l_j$ the inf.

---

*I.e. the complement allows no schlicht, bounded conformal mapping.*
of the length of all curves joining the two distinguished sides in $R_j'$, by $F_j$ the area of the open $R_j'$. Because $E_y$ is of measure zero and $\sum_i a_i \geq l$, it is clear that, given any $y \in B$, we can choose the $\alpha_j$ and afterwards $y_1 < y < y_2$ in such a way that

$$\sum_j a_j \leq \epsilon \quad \text{and} \quad \sum l_j \geq l - \epsilon.$$ 

The intervals $y_1 < y < y_2$ provide us with an open covering of $B$ from which we get a finite covering. As in Lemma 2 the restriction to distinct strips $S_i$ does not change the above two conditions, and we have for any strip $S_i$ (with its intervals $\epsilon_{ij}$ of length $a_{ij}$ and the corresponding rectangles $R_{ij}$ with moduli $m_{ij}$) the following estimates:

$$\frac{a_{ij}}{b_i} = m_{ij} \geq \frac{1}{K} \frac{1}{m_{ij}} \geq \frac{1}{K} \frac{\epsilon_{ij}^2}{F_{ij}}.$$ 

Therefore

$$\frac{\epsilon}{b_i} \geq \sum_j a_{ij}/b_i \geq \frac{1}{K} \sum_j \frac{\epsilon_{ij}^2}{F_{ij}} \geq \frac{1}{K} \left( \sum_j \frac{\epsilon_{ij}}{j} \right)^2 / \sum_i F_{ij} \geq \frac{1}{K} \frac{(l - \epsilon)^2}{\sum_i F_{ij}}.$$ 

Taking reciprocals and summing up we get

$$\frac{h}{\epsilon} \leq \sum_i b_i/\epsilon \leq K \sum_i F_{ij}/(l - \epsilon)^2 \leq K \frac{a'b'}{(l - \epsilon)^2}.$$ 

As $\epsilon \to 0$ we conclude $h = 0$, q.e.d.

With exactly the same method but less rough estimates we can get the following result:

**Lemma 3'.** Let $E$ be any closed set in $R$, $B$ a closed subset of $0 \leq y \leq b$ of measure $h$ and such that for $y \in B$ the linear measure of $E_y$ is $\leq l$ while the linear measure of $E'_y$ is $L(y) \geq l'$. Let $F$ denote the area of the closed set $\bigcup_{y \in B} E'_y$. Then we have

$$l/h \geq K^{-1}(l'/F).$$

For $l = 0$, $l' > 0$ we get $h = 0$, i.e. the above theorem. For $l' = \infty$ (we have to replace $l' - \epsilon$ by a number $< l'$ in the proof) and $l = a$ we get $h = 0$.

**Corollary.** The set of values $y$ for which the image curve of the segment $\exists z = y$ is not rectifiable is of measure zero.
Theorem 3. If $E$ is a closed subset of $G$ of two-dimensional measure zero and $w(z)$ any quasiconformal mapping of $G$, its maximal dilation on $G - E$ is equal to its maximal dilation on $G$. In other words, a closed set of zero area is deletable with respect to all quasiconformal mappings of $G$.

Proof. We consider any analytic quadrilateral $Q \subset G$ and map it as well as its image $Q'$ conformally onto the rectangles $R$ and $R'$ respectively. As the set $E_1$ in $R$ which corresponds to the part $E \cap Q$ of the exceptional set $E$ in $G$ is closed and of zero area, the conditions of Lemma 3 are fulfilled. From Lemma 3 we conclude that the conditions of Lemma 2 with the constant $K_0$ are fulfilled. From Lemma 2 we get therefore

$$a'/b' \leq K_0(a/b)$$

q.e.d.

5. Another application of the same method leads to the following Lemma 3’’: A topological mapping of $R$ onto $R'$ with the property (1) of Lemma 3 is absolutely continuous on almost every horizontal line.

The set of all $y$ for which the length $L_y(\xi)$ of the image of the stretch $(0 = x = \xi, y = \eta)$ is not absolutely continuous in $\xi$ is measurable. If it were not of measure zero, there would exist a closed set $B$ on the interval $0 \leq y \leq b$ of positive linear measure $\delta$ and a number $l$ such that for every $y$ in $B$ the corresponding horizontal stretch carries a system of intervals of total length $< \varepsilon$ while their images have total length $\geq l$. The rest of the proof is a repetition of the one given for Lemma 3. Lemma 3’’ proves the theorem on absolute continuity of quasiconformal mappings which was announced in the Bull. Amer. Math. Soc. Abstract 61-3-421, by the same author.

References


Institute for Advanced Study