ON MODIFIED BOREL METHODS

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1. Introduction. Given a series \( \sum a_n \) with partial sums \( s_n \) it is possible to associate with it the Borel transforms

\[
B(x; s_k) = e^{-x} \sum \frac{s_k x^k}{k!}, \quad B'(x; s_k) = \int_0^x e^{-t} a(t) dt, \quad a(t) = \sum \frac{a_k t^k}{k!}
\]

for \( x > 0 \). One says that \( B\)-lim \( s_n = s \) \( \Rightarrow B'\)-lim \( s_n = s \) if \( \lim_{x \to \infty} B(x; s_k) = s \) \( \Rightarrow \lim_{x \to \infty} B'(x; s_k) = s \). The relations between these Borel methods \( B, B' \), and their behavior under change of index are known \([8, p. 183; 6; 7]\).

Following a suggestion of R. P. Boas, Jr., we intend to study in this paper the modified Borel methods which arise when the continuous variable \( x \) in (1.1) is replaced by the discrete sequence of integers \( n = 1, 2, \cdots \). The resulting methods shall be denoted by \( B_I \) and \( B'_I \), and our interest is to discuss the relations among the methods \( B, B_I, B', B'_I \) (which is done in §3) and the behavior of these methods under change of index (cf. §4). The methods \( B_I, B'_I \) show certain abnormalities in comparison with \( B, B' \). For example, \( B\)-lim \( s_n = s \) always implies \( B'\)-lim \( s_n = s \), whereas \( B_I\)-lim \( s_n = s \) implies \( B'_I\)-lim \( s_n = s \) if \( a_n = O(K^n) \) for \( K < (\pi^2 + 1)^{1/2} \) and not for \( K = (\pi^2 + 1)^{1/2} \).

Our results are based on two theorems on entire functions (§2). The first allows one to infer \( f(x) \to s \) \( [x \to + \infty] \) from \( f(n) \to s \) \( (n = 1, 2, \cdots) \) if the type of \( f(z) \) is less than \( \pi \), and is well known; the second allows one to infer \( f(x) \equiv s e^x \) \( [x \to + \infty] \) from \( f(n) \equiv s e^n \) \( (n = 1, 2, \cdots) \) if the type of \( f(z) \) is less than \( (\pi^2 + 1)^{1/2} \).

Finally, in §5 Cesàro-Borel methods are considered but the results there are incomplete, whereas the results in §3 and §4 are in a certain sense best possible.

2. A theorem on functions of exponential type. If \( f(z) \) is regular in the angle \( |\arg z| \leq \alpha (\alpha > 0) \), it is said to be there of exponential type \( \tau \) if for every \( \epsilon > 0 \), but for no \( \epsilon < 0 \), there exists a constant \( M(\epsilon) \) such that

\[
|f(z)| \leq M(\epsilon) e^{(\tau + \epsilon)|z|} \quad (|\arg z| \leq \alpha).
\]

The growth of \( f(z) \) along the ray \( \arg z = \phi \) \( (|\phi| \leq \alpha) \) is described by the indicator function

Received by the editors January 14, 1955.
$$h_f(\phi) = \limsup_{r \to \infty} r^{-1} \log |f(re^{i\phi})|.$$ 

In §3 and §4 we meet the problem of going from the behavior of $f(n)$ ($n = 1, 2, \cdots$) to the behavior of $f(x)$ ($x \to +\infty$). A well known theorem in this direction is\footnote{Theorem 1 is implicitly contained in Cartwright [2], explicitly in Macintyre [9, p. 16]. See also Pfuger [12, pp. 312–314], Duffin-Schaefffer [5, pp. 142–143] and Boas [1, p. 180].}

**Theorem 1.** If $f(z)$ is regular and of exponential type in $|\arg z| \leq \alpha \leq \pi/2$ ($\alpha > 0$), and if

$$h_f(\pm \alpha) < \pi \sin \alpha,$$

then $f(n) \to 1$ ($n = 1, 2, \cdots$) implies $f(x) \to 1$ ($x \to +\infty$).

For our purposes we need an extension of Theorem 1 covering the case $f(n) \equiv e^n$ instead of $f(n) \to 1$.

**Theorem 2.** If $f(z)$ is regular and of exponential type in $|\arg z| \leq \alpha \leq \pi/2$ ($\alpha > 0$), and if

$$(2.1) \quad h_f(\pm \alpha) < \pi \sin \alpha + a \cos \alpha \quad (a \geq 0),$$

then

$$(2.2) \quad f(n) \equiv n^k e^{an} \ (n = 1, 2, \cdots) \text{ implies } f(x) \equiv x^k e^{ax} \ (x \to +\infty) \ (k \geq 0).$$

In particular, (2.2) is true if $f(z)$ is regular and of exponential type $\tau < (\pi^2 + a^2)^{1/2}$ in $\Re(z) \geq 0$; for $\tau = (\pi^2 + a^2)^{1/2}$ this is false.

**Proof.** Consider $g(z) = f(z)e^{-az}(z+1)^{-k}$ in $|\arg z| \leq \alpha$, where $(z+1)^k$ is assumed to be $>0$ for $z \geq 0$. For the indicator function of $g(z)$ on $\arg z = \pm \alpha$ we have

$$h_g(\pm \alpha) = \limsup_{r \to \infty} r^{-1} \log |g(re^{\pm i\alpha})|$$

$$= \limsup_{r \to \infty} r^{-1} \log |f(re^{\pm i\alpha})| - a \cos \alpha < \pi \sin \alpha$$

by (2.1), and hence, by Theorem 1, $g(n) \to 1$ ($n = 1, 2, \cdots$) implies $g(x) \to 1$ ($x \to +\infty$).

If, in particular, $f(z)$ is of exponential type $\tau < (\pi^2 + a^2)^{1/2}$ in $\Re(z) \geq 0$, we choose $\alpha$ such that $\tan \alpha = \pi/a$, so that

$$\pi \sin \alpha + a \cos \alpha = \pi/\sin \alpha = (\pi^2 + a^2)^{1/2} > \tau \geq h_f(\pm \alpha),$$

i.e. hypothesis (2.1) is fulfilled and hence (2.2) follows.

For the last part of the theorem consider $f(z) = e^{az}(\sin \pi z + 1)$.
3. Relations between the Borel methods. Now we are going to consider the methods of summability which associate with a given series the following transformations:

\[ B: \quad e^{-z} \sum \frac{s_k x^k}{k!} (x > 0); \quad B': \quad \int_0^z e^{-t} a(t) dt (x > 0); \]

\[ a(t) = \sum \frac{a_k t^k}{k!}, \]

\[ B_I: \quad e^{-z} \sum \frac{s_k n^k}{k!} (n = 1, 2, \cdots); \quad B'_I: \quad \int_0^z e^{-t} a(t) dt (n = 1, 2, \cdots). \]

The \( B \)- and \( B' \)-transformations are connected by the formal relation (Hardy \cite{8}, p. 182) \[ B(x; s_k) = B(x; a_k) + B'(x; s_k), \]

\[ (3.1) \quad \sum \frac{s_k x^k}{k!} = \sum \frac{a_k x^k}{k!} + \int_0^z e^{-t} a(t) dt. \]

The problem of this paragraph is to investigate the relative strength of the above Borel methods. For two summability methods \( V_1 \) and \( V_2 \) we use the notation \( V_1 \rightarrow V_2 \), if \( V_1 \)-lim \( s_n = s \) implies always \( V_2 \)-lim \( s_n = s \).

The following relations are trivial or known.

\( (3.2) B \rightarrow B_I \) and \( B' \rightarrow B'_I \).

\( (3.3) B \rightarrow B' \) (Hardy \cite{8}, p. 183). \[ B(x; s_k) = B(x; a_k) + B'(x; s_k), \]

\( (3.4) B' \rightarrow B \) if \( a_n = O(K^n) \) for some \( K > 0 \) (Gaier \cite{6}, p. 455). This becomes false if \( a_n = O(K^n) \) is replaced by \( a_n = O(n^n K^n) \) (\( e \) arbitrary >0) (Gaier \cite{7}).

Our new results about the relations between the Borel methods are summarized in

**Theorem 3.** (1) \( B_I \rightarrow B \), \( B_I \rightarrow B' \), and \( B_I \rightarrow B'_I \), if \( a_n = O(K^n) \) for \( K < (\pi^2 + 1)^{1/2} \), but not for \( K = (\pi^2 + 1)^{1/2} \).

(2) \( B'_I \rightarrow B' \), \( B'_I \rightarrow B \), and \( B'_I \rightarrow B_I \), if \( a_n = O(K^n) \) for \( K < (\pi^2 + 1)^{1/2} \), but not for \( K = (\pi^2 + 1)^{1/2} \).

Note, in particular, that there is no analogy to (3.3) for the methods \( B_I \) and \( B'_I \).

**Proof.** (1) (a) \( B_I \rightarrow B \). (i) If \( a_n = O(K^n) \) \( (K < (\pi^2 + 1)^{1/2}) \), then \( |s_n| \leq MK'^n \) \( (K' < (\pi^2 + 1)^{1/2}) \) and the entire function \( \phi(z) = \sum s_n z^n / n! \) satisfies the estimation

\[ |\phi(z)| \leq M \sum \frac{K'^n |z|^n}{n!} = Me^{K'|z|}, \]
i.e. it is of type $\tau < (\pi^2 + 1)^{1/2}$. Therefore the assumption $\phi(n) \leq A \cdot e^n$ ($n = 1, 2, \cdots$) implies, by Theorem 2, $\phi(x) \leq A \cdot e^x$ ($x \to +\infty$), i.e. $B$-lim $s_n = A$.

(ii) Define $s_n$ by $\sum (s_n x^n / n!) = e^x (\sin \pi x + 1)$. Then $(\alpha) B_I$-lim $s_n = 1$, but not $B$-lim $s_n = 1$. (\beta) One finds immediately

$$s_n = 1 + (1/2i) \{(1 + i\pi)^n - (1 - i\pi)^n\},$$

so that $s_n = O((\pi^2 + 1)^{n/2})$ and also $a_n = O((\pi^2 + 1)^{n/2})$ are fulfilled.

(b) $B_I \to B'$. (i) The assumption about the $a_n$ implies (Case (a) and (3.3))

$$B_I \to B \to B'.$$

(ii) Define $a_n$ by

$$\int_0^1 e^{-t a(t)} dt = \sin (\pi z + \alpha); \quad t\alpha = -\pi.$$

Then $(\alpha)$ $B_I$-lim $s_n = 0$. For, by the relation (3.1), we have

$$B(x; s_k) = \frac{d}{dx} \sin (\pi x + \alpha) + \sin (\pi x + \alpha),$$

which, taken at $x = n$ ($n = 1, 2, \cdots$), becomes

$$B(n; s_k) = \cos \pi n (\sin \alpha + \pi \cos \alpha) = 0 \quad (n = 1, 2, \cdots).$$

On the other hand $B'_I$-lim $s_n$ does not exist. (\beta) We have

$$a(t) = e^t \cdot \pi \cos (\pi t + \alpha) = \sum \frac{a_k t^k}{k!},$$

from which $a_n = O((\pi^2 + 1)^{n/2})$ is immediate.

(2) (a) $B'_I \to B'$. (i) If $a_n = O(K^n)$ ($K < (\pi^2 + 1)^{1/2}$), then $a(t)$ is an entire function of exponential type $\tau < (\pi^2 + 1)^{1/2}$. If therefore $g(z) = e^{-i\alpha} g(z)$, we have for the indicator function of $g(z)$ taken for the rays $\arg z = \pm \alpha$ ($t\alpha = \pi$)

$$h_0(\pm \alpha) = h_0(\pm \alpha) - \cos \alpha < (\pi^2 + 1)^{1/2} - \cos \alpha = \pi \sin \alpha,$$
and hence for the function $\phi(z) = \int_0^z e^{-t^a(t)} dt$

$$h_\phi(\pm \alpha) < \pi \sin \alpha,$$

so that an application of Theorem 1 infers $\phi(x) \to A(x \to +\infty)$ from $\phi(n) \to A (n = 1, 2, \cdots)$.

(ii) Define $a_n$ by $\int_0^z e^{-t^a(t)} dt = \sin \pi z$. Obviously $B'_f -\lim s_n = 0$, but not $B' -\lim s_n = 0$. The validity of $a_n = O((\pi^2 + 1)^n/2)$ is again immediate.

(b) $B'_f \to B$. (i) The assumption about the $a_n$ implies (Case (a) and (3.4))

$$B'_f \to B' \to B.$$

(ii) Define $a_n$ as in (2) (a). $B$-lim $s_n = 0$ cannot hold since $B' -\lim s_n$ does not exist.

(c) $B_I \to B_I$. (i) By the preceding case $B'_f \to B \to B_I$.

(ii) Define $a_n$ as in (2) (a). By (3.1), the $B$-transform of the corresponding sequence $s_n$ is $\sin \pi x + \pi \cos \pi x$, so that $B_I(n; s_k) = \pm \pi (n = 1, 2, \cdots)$.

4. On the change of index for the methods $B_I$ and $B'_f$. We consider the two series

$$\sum a_k = a_0 + a_1 + a_2 + \cdots$$

with partial sums $s_n$ and

$$\sum b_k = 0 + a_0 + a_1 + \cdots$$

with partial sums $t_n$.

The problem is to determine under what conditions

(4.1.a) $V\text{-lim } s_n = s$ implies $V\text{-lim } t_n = s$

or

(4.1.b) $V\text{-lim } t_n = s$ implies $V\text{-lim } s_n = s$,

where $V$ is one of the methods $B_I, B'_f$.

Note that $B(x; b_k) = (d/dx)B'(x; t_k)$. The proof of (4.3) follows from

$$B'(x; t_k) = \int_0^x e^{-t b(t)} dt = - e^{-t b(t)} \bigg|_0^x + \int_0^x e^{-t a(t)} dt = - B(x; b_k) + B'(x; s_k).$$

**Theorem 4.** If $V$ is one of the methods $B_1, B_1'$, both statements (4.1.a) and (4.1.b) are correct if $a_n = O(K^n)$ for $K < (\pi^2 + 1)^{1/2}$, but not for $K = (\pi^2 + 1)^{1/2}$.

Note, in particular, that there is no analogy to the fact that (4.1.a) holds for $V = B$ without restriction of the $a_n$.

**Proof.** (1) $V = B_1$. (a) By (4.2), $B_1$-lim $t_n = s$ if and only if $B_1'$-lim $s_n = s$, which follows from $B_1$-lim $s_n = s$ if $a_n = O(K^n)$ for $K < (\pi^2 + 1)^{1/2}$, but not for $K = (\pi^2 + 1)^{1/2}$ (Theorem 3, 1c).

(b) Again, $B_1$-lim $t_n = s$ if and only if $B_1'$-lim $s_n = s$, which implies $B_1$-lim $s_n = s$ if $a_n = O(K^n)$ for $K < (\pi^2 + 1)^{1/2}$, but not for $K = (\pi^2 + 1)^{1/2}$ (Theorem 3, 2c).

(2) $V = B_1'$. (a) (i) If $a_n = O(K^n)(K < (\pi^2 + 1)^{1/2})$, $B_1'$-lim $s_n = s$ implies $B_1'$-lim $s_n = s$ so that by (4.3) $\phi(x) + \phi'(x) \rightarrow s(x \rightarrow + \infty)$ $[\phi'(x) = B(x; b_k)]$, and consequently (Hardy [8, p. 107]) $\phi(x) \rightarrow s(x \rightarrow + \infty)$, i.e. $B_1'$-lim $t_n = s$.

(ii) Define $b_n$ by $\int_0^b e^{-t b(t)} dt = \sin (\pi x + \alpha)$ with $tg\alpha = - \pi$ and proceed as in Theorem 3, 1(c) (ii). We get $B_1'$-lim $s_n = 0$ whereas $B_1'$-lim $t_n$ does not exist, although $a_n = O((\pi^2 + 1)^{1/2})$.

(b) (i) If $a_n = O(K^n)(K < (\pi^2 + 1)^{1/2})$, $B_1'$-lim $t_n = s$ implies $B_1'$-lim $t_n = s$ (Theorem 2), and since $B'(z; t_k)$ is an entire function of exponential type tending to $s$ as $z \rightarrow + \infty$, its derivative $e^{-t b'(z)} = B'(z; b_k)$ tends to zero as $z \rightarrow + \infty$ (Boas [1, p. 212] and Gaier [6, p. 454]) which, by (4.3), implies $B_1'$-lim $s_n = s$.

(ii) Define $b_n$ by $\int_0^b e^{-t b(t)} dt = \sin \pi x$. Then $B_1'$-lim $t_n = 0$, but not $B_1'$-lim $s_n = 0$, although $a_n = O((\pi^2 + 1)^{n/2})$.

5. Cesàro-Borel methods. Doetsch [3] was the first to consider the Cesàro-Borel transform

$$C_k B(x; s_k) = k x^{-k} \int_0^x B(t; s_k)(x-t)^{k-1} dt \quad (k > 0, x \geq 0),$$

and in view of our results in §3 one can ask what relations there are

* Let $b(t) = \sum (b_n t^n / n!)$, so that $b'(t) = a(t)$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
for example between the methods $C_kB$ and $C_kB_I$ ($C_k =$ matrix method in the latter case). It is not surprising that in general

$$\text{(5.1)} \quad C_kB_I\text{-lim } s_n = s \text{ does not imply } C_kB\text{-lim } s_n = s;$$

however, also

$$\text{(5.2)} \quad C_kB\text{-lim } s_n = s \text{ does not imply } C_kB_I\text{-lim } s_n = s.$$  

Equivalent to the problem raised is, of course, under what conditions for an entire function $f(z)$ does

$$C_k\text{-lim } f(n) = s \text{ imply } C_k\text{-lim } f(x) = s$$

and conversely. For $k = 1$ the statement (5.1) follows from consideration of $f(z) = z \sin \pi z$, whereas for the proof of (5.2) we take an entire function $f(z)$ of exponential type $(< 2\pi + \epsilon)$ which is, for $x > 0$,

$$f(x) = x^{1/2} \cos 2\pi x + o(1).$$

Then obviously $C_1\text{-lim } f(n) = +\infty$, but $C_1\text{-lim } f(x) = 0$. The author has no contribution towards the solution of this problem.

**References**


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Note: To obtain such a function apply Macintyre's lemma [1, p. 80] to $f(z) = z^{1/8} \cdot \cos 2\pi z; f(z/2 + \epsilon)$ is of type $< \pi$ in $\Re(z) \geq 0$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use