REAL LINEAR CHARACTERS OF THE SYMPLECTIC MODULAR GROUP

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1. The symplectic modular group \( \Gamma_{2n} \) consists of all integral \( 2n \times 2n \) matrices \( \mathbf{M} \) for which \( \mathbf{M} \mathbf{f} \mathbf{M}' = \mathbf{f} \), where

\[
\mathbf{f} = \begin{pmatrix} 0 & I^{(n)} \\ -I^{(n)} & 0 \end{pmatrix}.
\]

In order to determine all possible automorphisms of \( \Gamma_{2n} \) [1], it is necessary to find all real linear characters of \( \Gamma_{2n} \), that is, all homomorphisms into \( \{ \pm 1 \} \). In this note we prove that \( \Gamma_{2n} \) has no nontrivial real linear characters for \( n > 2 \), while \( \Gamma_2 \) and \( \Gamma_4 \) each have exactly one nontrivial real linear character. We shall also determine \( \Gamma_{2n}' \), the commutator subgroup of \( \Gamma_{2n} \).

We define the symplectic direct sum \( \mathfrak{M}_1 \star \mathfrak{M}_2 \) by

\[
\mathfrak{M}_1 \star \mathfrak{M}_2 = (A_1 B_1) \star (A_2 B_2) = \begin{pmatrix}
A_1 & B_1 & 0 \\
0 & A_2 & B_2 \\
C_1 & D_1 & 0 \\
0 & C_2 & D_2
\end{pmatrix}.
\]

Set

\[
S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

and define\(^2\)

\[
U_0 = S + I^{(n-2)}, \quad U_1 = V + I^{(n-2)}, \quad U_2 = T + I^{(n-2)}.
\]

Then \([2]\) \( \Gamma_{2n} \) is generated by \( \mathfrak{R}_i = U_i + U_i^{-1} \) (\( i = 0, 2 \)), \( \mathfrak{T}_0 = T \star I^{2(n-1)} \), \( \mathfrak{S}_0 = S \star I^{2(n-1)} \), and their conjugates. When \( n = 1 \), the \( \mathfrak{R}_i \) are superfluous. Next we remark that

\[
\begin{align*}
(1) & \quad \mathfrak{S}_0 \mathfrak{T}_0 = (ST) \star I = (ST)^{-2} \star I, \\
(2) & \quad \mathfrak{R}_0 \mathfrak{R}_2 = U_3 + U_3^{-1}, \text{ where } U_3 = ST + I = (ST + I)^{-2}, \\
(3) & \quad \mathfrak{T}_0 = \mathfrak{R}_1 \mathfrak{R}_2 \cdot \mathfrak{R}_0 \mathfrak{T}_0 \mathfrak{R}_0^{-1} \mathfrak{T}_0^{-1} \cdot (\mathfrak{R}_1 \mathfrak{R}_2)^{-1} \cdot \mathfrak{S}_0 \mathfrak{R}_1 \cdot \mathfrak{R}_2 \cdot (\mathfrak{S}_0 \mathfrak{R}_1)^{-1}.
\end{align*}
\]

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1 Numbers in brackets refer to the bibliography at the end of the paper.
2 \( A \star B \) denotes the direct sum of the matrices \( A \) and \( B \).
Therefore if \( \theta \) is any real linear character of \( \Gamma_{2n} \), we must have

\[
\theta(R_0) = \theta(R_2) = \theta(T_0) = \theta(S_0) = \pm 1.
\]

On the other hand, let \( \Omega_n \) be the unimodular group consisting of all integral \( n \times n \) matrices with determinant \( \pm 1 \). Then for \( n \geq 2 \), \( \Omega_n \) is its own commutator subgroup [3]. Hence for \( n \geq 2 \), \( U_0 \) is a product of commutators in \( \Omega_n \), and therefore \( R_0 \) is in the commutator subgroup of \( \Gamma_{2n} \). Therefore \( \theta(R_0) = +1 \), so \( \Gamma_{2n} \) has no nontrivial real linear characters for \( n \geq 2 \).

Now we must prove that there exists a homomorphism of \( \Gamma_{2n} \) into \( \{ \pm 1 \} \) which maps each generator \( R_0, R_2, T_0, S_0 \) into \(-1\), for the cases \( n = 1 \) and \( n = 2 \). This is already known for \( n = 1 \) [3], but we give an independent proof here. Let \( H \) be the normal subgroup of \( \Gamma_{2n} \) consisting of all matrices \( \equiv I^{(2n)} \) (mod 2). Then it is known [4] that \( \Gamma_{2n}/H \cong S_{2n} \) for \( n = 1, 2 \), where \( S_k \) is the symmetric group on \( k \) symbols. Let \( \pi \) be the homomorphism mapping \( \Gamma_{2n} \) onto \( S_{2n} \), and let \( A_{2n} \) be the alternating subgroup of \( S_{2n} \). Then \( \pi^{-1}(A_{2n}) \) is a subgroup of index 2 of \( \Gamma_{2n} \), \( n = 1, 2 \). Therefore \( \Gamma_{2n} \) has a nontrivial real linear character for \( n = 1, 2 \), and the previous discussion shows that it is unique, and maps each generator onto \(-1\).

2. Now we consider \( \Gamma'_{2n} \), and we begin with \( n = 1 \), the most difficult case. The commutator subgroup of \( \Gamma_2/\{ \pm I \} \) is known [5], but we shall not use this earlier result. According to [6], \( \Gamma_2 = \{ S, T \} \) has as defining relations

\[
S^4 = TS^{-1}TS^{-1}TS = 1.
\]

Then the sum of the exponents to which \( S \) (resp. \( T \)) occurs in any relation, must be of the form \( 4a - b \) (resp. \( 3b \)), where \( a \) and \( b \) are integers. For \( X \in \Gamma_2 \) let \( \alpha_X = \) sum of the exponents to which \( S \) occurs, and let \( \beta_X = \) sum of the exponents to which \( T \) occurs, when \( X \) is expressed as a power product of \( S \) and \( T \). Then \( X \in \Gamma'_2 \) implies that \( \alpha_X, \beta_X \) are of the form

\[
\alpha_X = 4a - b, \quad \beta_X = 3b,
\]

for integral \( a, b \). On the other hand,

\[
S^{4a-b}T^{3b} = S^{-b}T^{3b} \equiv (S^{-1}T^3)^b \pmod{\Gamma'_2},
\]

and

\[
S^{-1}T^3 = S^{-1}T \cdot ST^{-1}ST^{-1}S^{-1} \cdot T \in \Gamma'_2.
\]

Hence \( X \in \Gamma'_2 \) if and only if (4) holds. Consequently \( T^{12} \in \Gamma'_2, T^m \in \Gamma'_2 \) for \( m = 1, \cdots, 11 \), and we have
\[ \Gamma_2 = \bigcup_{m=0}^{11} T^m \Gamma_2'. \]

Thus \((\Gamma_2: \Gamma_2') = 12.\)

Next we show how \(\Gamma_2'\) may be defined by means of congruences. Let

\[ H_m = \{ X \in \Gamma_2 : X \equiv I \pmod{m} \}. \]

Then \(H_m\) is a normal subgroup of \(\Gamma_2\), and in particular \(\Gamma_2/H_3\) is a group of order 24 consisting of all 2x2 matrices of determinant +1 with elements in \(GF(3)\). This group contains a normal subgroup \([C, D]\) \[4\] of index 3, where

\[ C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}. \]

Hence the group \(K_3\) consisting of all elements of \(\Gamma_2\) congruent (mod 3) to a matrix in \([C, D]\) is a normal subgroup of \(\Gamma_2\) of index 3. Therefore \(\Gamma_2' \subseteq K_3.\)

Next we remark that \(\Gamma_2/H_4\) is of order 48, and contains the normal subgroup \([A, E, F]\) of order 12 generated by

\[ A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \]

taken mod 4. If \(K_4\) is the set of all elements of \(\Gamma_2\) congruent mod 4 to a matrix in \([A, E, F]\), then \(K_4\) is a normal subgroup of \(\Gamma_2\) of index 4. Therefore \(\Gamma_2' \subseteq K_4.\) Since \(K_4K_3 = \Gamma_2,\) it follows at once that

\((\Gamma_2' : K_3 \cap K_4) = 12,\)

and so

\[ \Gamma_2' = K_3 \cap K_4. \]

3. We show next that \((\Gamma_4 : \Gamma_4') = 2,\) and also we determine \(\Gamma_4'\) by means of congruences. From \([3]\) we find that \(R_0R_2\) and \(R_2^2 \in \Gamma_4'.\) Hence \(L = \Gamma_4' \cup R_2\Gamma_4'\) is a normal subgroup of \(\Gamma_4,\) and (using \((3)) \(R_0, R_2,\) and \(\Sigma_0\) are elements of \(L.\) Also we have

\[ \Sigma_0^2 \Sigma_0 = \Sigma_0 \Sigma_0^{-1} (\Sigma_0 \Sigma_0^{-1})^2 \]

so \(\Sigma_0 \in L.\) Hence \(L = \Gamma_4,\) and therefore either \(\Gamma_4' = \Gamma_4\) or \((\Gamma_4 : \Gamma_4') = 2.\)

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3 This result has been obtained independently by Professor J. L. Brenner.

4 The author wishes to acknowledge with thanks some helpful conversations with Professor E. V. Schenkman on the material in \(\S 2.\)
However we have already seen that $\Gamma_4$ contains a subgroup $K$ of index 2. Since $\Gamma_4' \subset K$, we then have $\Gamma_4' = K$.

Finally we remark that the previous discussion shows easily that $\Gamma_{2n}' = \Gamma_{2n}$ for $n > 2$.

**BIBLIOGRAPHY**