FIELD CONVEXITY OF A SQUARE MATRIX

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1. **Introduction.** Let C be a complex n-square matrix and z a complex n-vector. Let (u, v) be the unitary inner product and set $f(z) = (Cz, z)$. The range of values of f for $\|z\| = (z, z)^{1/2} = 1$ is the *field of values* [1] of C and we denote it by $F(C)$. Set $A = (C + C^*)/2$, $B = (C - C^*)/2i$, $\phi(z) = (Az, z)$, and $\psi(z) = (Bz, z)$. Then $f(z) = \phi(z) + i\psi(z)$ and since A and B are Hermitian, ϕ and ψ are real-valued.

It is known that $F(C)$ is convex in the plane [1]. To indicate briefly the proof of this fact note that $F(UCU^*) = F(C)$ for U a unitary matrix. Hence we may assume $\phi(z) = \sum_{i=1}^{n} \mu_i |z_i|^2$ where the μ_i are the real eigenvalues of A. It is easy to show that for $\|z_0\| = 1$, $\phi^{-1}(\phi(z_0))$ is connected on the unit sphere and thus that $\psi(\phi^{-1}(\phi(z_0)))$ is connected in the plane. This implies that $f(\phi^{-1}(\phi(z_0)))$ is an interval and since convexity is unaltered by rotation we may conclude that every line intersects $F(C)$ in an interval.

In case C is normal there exists a unitary matrix diagonalizing C and hence we may assume

$$f(z) = \sum_{i=1}^{n} \lambda_i |z_i|^2. \tag{1.1}$$

Thus $F(C)$ is the convex polygon spanned by the eigenvalues of C. For any n-square matrix C let $P(C)$ be the polygon spanned by the eigenvalues and let Q^n be the set of complex n-square matrices such that $P(C) = F(C)$. Also let N^n be the set of all complex n-square normal matrices. We show that for $n \leqq 4$, $Q^n = N^n$; but for $n \geqq 5$, $N^n \subset Q^n$ and $Q^n \neq N^n$. However, we state necessary and sufficient conditions that $P(C) = F(C)$ for C triangular and n arbitrary. By Schur's Lemma [2] any matrix is unitarily similar to a triangular matrix and hence these conditions may be applied when C is arbitrary by first reducing to triangular form.

2. **Results.**

THEOREM 1. *If C is triangular and $C \in Q^n$ then, for $p < q$, $c_{pq} = 0$ if either c_{pp} or c_{qq} is on the boundary of $P(C)$.*

PROOF. C triangular implies

$$f(z) = \sum_{q=1}^{n} \lambda_q |z_q|^2 + \sum_{1=p<q}^{n} c_{pq}\bar{z}_p z_q \tag{2.1}$$

Presented to the Society, December 29, 1954; received by the editors November 22, 1954.

where $\lambda_q = c_{qq}$ are the eigenvalues of C. Let λ_{p_α}, λ_{p_β} be successive vertices of $P(C)$ taken in a counter clockwise sense. To see if $F(C)$ can lie on both sides of the segment joining λ_{p_α} and λ_{p_β} we investigate the relationship between $R_\alpha(z) = \text{Re } (\exp(-i\theta_\alpha)f(z))$ and $\lambda_{\alpha p_\alpha}$, where $\lambda_{\alpha q} = \text{Re } (\exp(-i\theta_\alpha)\lambda_q)$ and $\theta_\alpha = \arg (\lambda_{p_\beta} - \lambda_{p_\alpha}) - \pi/2$. Set $|z_p| = s_p$, $|c_{pq}| = r_{pq}$ and $\phi(p, q, \alpha) = \arg c_{pq} - \arg z_p + \arg z_q - \theta_\alpha$. Then

$$
\begin{aligned}
R_\alpha(z) &= \sum_{q=1}^{n} \lambda_{\alpha q} s_q^2 + \sum_{p<q} \cos \phi(p, q, \alpha) r_{pq} s_p s_q \\
&= \lambda_{\alpha p_\alpha} - \sum_{q=1}^{n} (\lambda_{\alpha p_\alpha} - \lambda_{\alpha q}) s_q^2 + \sum_{p<q} \cos \phi(p, q, \alpha) r_{pq} s_p s_q,
\end{aligned} \tag{2.2}
$$

since $\sum_{q=1}^{n} s_q^2 = 1$. If $r_{p_0 q_0} \neq 0$, where λ_{p_0} is any eigenvalue on the segment joining λ_{p_α} and λ_{p_β}, set $z_q = 0$ for $q \neq q_0$ or p_0, $\arg z_{p_0} = \arg c_{p_0 q_0} + \arg z_{q_0} - \theta_\alpha$, and

$$
s_{p_0} > \frac{(\lambda_{\alpha p_\alpha} - \lambda_{\alpha q_0})}{r_{p_0 q_0}} s_{q_0}.
$$

For such a vector z

$$
R_\alpha(z) > \lambda_{\alpha p_\alpha} = \lambda_{\alpha p_0}
$$

and we conclude that $C \notin Q^n$. Similarly $R_\alpha(z) > \lambda_{\alpha p_\alpha}$ for any $c_{q p_0} \neq 0$.

Corollary. *$Q^n = N^n$ for $n \leqq 4$.*

Proof. $N^n \subseteq Q^n$ follows from (1.1). Assume $A \in Q^n$ and transform A to triangular form C by a unitary matrix. For $n \leqq 4$ at most one eigenvalue of A lies in the interior of $P(A) = P(C)$ and, by Theorem 1, C is diagonal. Hence A is normal and $A \in N^n$.

Similarly we have the following

Corollary. *If at most one eigenvalue of A lies in the interior of $P(A)$ then $A \in Q^n$ implies $A \in N^n$.*

Denote by $\sum_p'$ a sum obtained by deleting those indices p for which c_{pp} is on the boundary of $P(C)$. Similarly $\sum_{p<q}'$ is the sum obtained by deleting those (p, q) for which at least one of p or q is such that c_{pp} or c_{qq} is on the boundary of $P(C)$. Set

$$
\begin{aligned}
d_{\alpha q} &= \lambda_{\alpha p_\alpha} - \lambda_{\alpha q}, \\
z_q &= x_q + i y_q, \\
c_{pq} &= t_{pq} + i u_{pq}.
\end{aligned} \tag{2.3}
$$

Also define $S_\alpha(z)$ by the formula

$$
\begin{aligned}
S_\alpha(z) = \lambda_{\alpha p_\alpha} &- \sum_q{}' d_{\alpha q}(x_q^2 + y_q^2) \\
&+ \sum_{p<q}{}' (t_{pq}\cos\theta_\alpha + u_{pq}\sin\theta_\alpha)(x_p x_q + y_p y_q) \\
&+ \sum_{p<q}{}' (t_{pq}\sin\theta_\alpha - u_{pq}\cos\theta_\alpha)(x_p y_q - y_p x_q).
\end{aligned}
\tag{2.4}
$$

THEOREM 2. *If C is triangular with k eigenvalues lying in the interior of $P(C)$, then $C\in Q^n$ if and only if:*

(a) *$c_{pq}=0$ for $p<q$, when either c_{pp} or c_{qq} is on the boundary of $P(C)$; and*

(b) *the $2k$ quadratic form $\lambda_{\alpha p_\alpha}-S_\alpha(z)$ is positive semidefinite for each value of α corresponding to a side of $P(C)$.*

PROOF. From (2.2) and Theorem 1 we have

$$
R_\alpha(z) \geqq \lambda_{\alpha p_\alpha} - \sum_q{}' d_{\alpha q}s_q^2 + \sum_{p<q}{}' \cos\phi(p, q, \alpha) r_{pq}s_p s_q, \tag{2.5}
$$

and equality holds for a suitable choice of z. The result follows by substituting the relations (2.3) in (2.5), expanding, and noting (2.4).

Condition (b) of Theorem 2 may be expressed in terms of the positive semidefiniteness of the symmetric matrices B_α associated with $\lambda_{\alpha p_\alpha}-S_\alpha(z)$. The application of the criterion thus amounts to an inspection of the variations in sign of the coefficients of the characteristic polynomials of the B_α.

In some cases it is possible to choose the vector z so that each $\cos\phi(p, q, \alpha)$ appearing in (2.5) is unity; then $\lambda_{\alpha p_\alpha}-S_\alpha(z)$ is a k-quadratic form in the s_q.

REFERENCES

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