FIELD CONVEXITY OF A SQUARE MATRIX

B. N. MOYLS AND M. D. MARCUS

1. Introduction. Let $C$ be a complex $n$-square matrix and $z$ a complex $n$-vector. Let $(u, v)$ be the unitary inner product and set $f(z) = (Cz, z)$. The range of values of $f$ for $||z||=(z, z)^{1/2}=1$ is the field of values [1] of $C$ and we denote it by $F(C)$. Set $A = (C + C^*)/2$, $B = (C - C^*)/2i$, $\phi(z) = (Az, z)$, and $\psi(z) = (Bz, z)$. Then $f(z) = \phi(z) + i\psi(z)$ and since $A$ and $B$ are Hermitian, $\phi$ and $\psi$ are real-valued.

It is known that $F(C)$ is convex in the plane [1]. To indicate briefly the proof of this fact note that $F(UCU^*) = F(C)$ for $U$ a unitary matrix. Hence we may assume $\phi(z) = \sum_{i=1}^{n} \mu_i |z_i|^2$ where the $\mu_i$ are the real eigenvalues of $A$. It is easy to show that for $||z_0|| = 1$, $\phi^{-1}(\phi(z_0))$ is connected on the unit sphere and thus that $\psi(\phi^{-1}(\phi(z_0)))$ is connected in the plane. This implies that $f(\phi^{-1}(\phi(z_0)))$ is an interval and since convexity is unaltered by rotation we may conclude that every line intersects $F(C)$ in an interval.

In case $C$ is normal there exists a unitary matrix diagonalizing $C$ and hence we may assume

$$
(1.1) \quad f(z) = \sum_{i=1}^{n} \lambda_i |z_i|^2.
$$

Thus $F(C)$ is the convex polygon spanned by the eigenvalues of $C$. For any $n$-square matrix $C$ let $P(C)$ be the polygon spanned by the eigenvalues and let $Q^n$ be the set of complex $n$-square matrices such that $P(C) = F(C)$. Also let $N^n$ be the set of all complex $n$-square normal matrices. We show that for $n \leq 4$, $Q^n = N^n$; but for $n \geq 5$, $N^n \subset Q^n$ and $Q^n \neq N^n$. However, we state necessary and sufficient conditions that $P(C) = F(C)$ for $C$ triangular and $n$ arbitrary. By Schur’s Lemma [2] any matrix is unitarily similar to a triangular matrix and hence these conditions may be applied when $C$ is arbitrary by first reducing to triangular form.

2. Results.

THEOREM 1. If $C$ is triangular and $C \in Q^n$ then, for $p < q$, $c_{pq} = 0$ if either $c_{pp}$ or $c_{qq}$ is on the boundary of $P(C)$.

PROOF. $C$ triangular implies

$$
(2.1) \quad f(z) = \sum_{q=1}^{n} \lambda_q |z_q|^2 + \sum_{1 < p < q} c_{pq} z_p z_q
$$

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where $\lambda_q = c_{qq}$ are the eigenvalues of $C$. Let $\lambda_{pA}$, $\lambda_{pB}$ be successive vertices of $P(C)$ taken in a counter clockwise sense. To see if $F(C)$ can lie on both sides of the segment joining $\lambda_{pA}$ and $\lambda_{pB}$ we investigate the relationship between 

$$R_{\alpha}(z) = \text{Re } (\exp(-i\theta_{\alpha})f(z))$$

and $\lambda_{\alpha pA}$, where

$$\lambda_{q} = \text{Re } (\exp(-i\theta_{q})\lambda_{q})$$

and $\theta_{q} = \text{arg } (\lambda_{pB} - \lambda_{pA}) - \pi/2$. Set $|z_p| = s_p$, $|c_{pq}| = r_{pq}$ and $\phi(p, q, \alpha) = \text{arg } c_{pq} - \text{arg } z_p + \text{arg } z_q - \theta_{\alpha}$. Then

$$R_{\alpha}(z) = \sum_{q=1}^{n} \lambda_{aq} s_q^2 + \sum_{p<q} \cos \phi(p, q, \alpha) r_{pq} s_p s_q$$

(2.2)

$$= \lambda_{\alpha pA} - \sum_{q=1}^{n} (\lambda_{\alpha pA} - \lambda_{aq}) s_q^2 + \sum_{p<q} \cos \phi(p, q, \alpha) r_{pq} s_p s_q,$$

since $\sum_{q=1}^{n} s_q^2 = 1$. If $r_{pq0} \neq 0$, where $\lambda_{pq}$ is any eigenvalue on the segment joining $\lambda_{pA}$ and $\lambda_{pB}$, set $z_q = 0$ for $q \neq q_0$ or $p_0$, $\text{arg } z_{pq} = \text{arg } c_{pq}$, and

$$s_{pq0} > \frac{(\lambda_{\alpha pA} - \lambda_{aq})}{r_{pq0}} s_{q0}.$$

For such a vector $z$

$$R_{\alpha}(z) > \lambda_{\alpha pA} = \lambda_{ap0}$$

and we conclude that $C \in Q^n$. Similarly $R_{\alpha}(z) > \lambda_{\alpha pA}$ for any $c_{pq} \neq 0$.

**Corollary.** $Q^n = N^n$ for $n \leq 4$.

**Proof.** $N^n \subseteq Q^n$ follows from (1.1). Assume $A \in Q^n$ and transform $A$ to triangular form $C$ by a unitary matrix. For $n \leq 4$ at most one eigenvalue of $A$ lies in the interior of $P(A) = P(C)$ and, by Theorem 1, $C$ is diagonal. Hence $A$ is normal and $A \in N^n$.

Similarly we have the following

**Corollary.** If at most one eigenvalue of $A$ lies in the interior of $P(A)$ then $A \in Q^n$ implies $A \in N^n$.

Denote by $\sum_{\rho}'$ a sum obtained by deleting those indices $\rho$ for which $c_{\rho\rho}$ is on the boundary of $P(C)$. Similarly $\sum_{\rho < q}'$ is the sum obtained by deleting those $(\rho, q)$ for which at least one of $\rho$ or $q$ is such that $c_{\rho\rho}$ or $c_{qq}$ is on the boundary of $P(C)$. Set

$$d_{\rho q} = \lambda_{\rho q} - \lambda_{\rho q},$$

(2.3)

$$z_q = x_q + iy_q,$$

$$c_{pq} = t_{pq} + i\mu_{pq}.$$

Also define $S_{\alpha}(z)$ by the formula
\[ S_\alpha(z) = \lambda_{\alpha p} - \sum_q' d_{aq}(x_q^2 + y_q^2) \]

\[
+ \sum_{p<q} (t_{pq} \cos \theta_a + u_{pq} \sin \theta_a)(x_px_q + y_py_q) \\
+ \sum_{p<q} (t_{pq} \sin \theta_a - u_{pq} \cos \theta_a)(x_py_q - y_px_p).
\]

**Theorem 2.** If \( C \) is triangular with \( k \) eigenvalues lying in the interior of \( P(C) \), then \( C \in \mathbb{Q}^n \) if and only if:

(a) \( c_{pq} = 0 \) for \( p < q \), when either \( c_{pp} \) or \( c_{qq} \) is on the boundary of \( P(C) \);

(b) the \( 2k \) quadratic form \( \lambda_{\alpha p} - S_\alpha(z) \) is positive semidefinite for each value of \( \alpha \) corresponding to a side of \( P(C) \).

**Proof.** From (2.2) and Theorem 1 we have

\[
R_\alpha(z) \geq \lambda_{\alpha p} - \sum_q' d_{aq}s_q^2 + \sum_{p<q} \cos \phi(p, q, \alpha)s_p s_q,
\]

and equality holds for a suitable choice of \( z \). The result follows by substituting the relations (2.3) in (2.5), expanding, and noting (2.4).

Condition (b) of Theorem 2 may be expressed in terms of the positive semidefiniteness of the symmetric matrices \( B_\alpha \) associated with \( \lambda_{\alpha p} - S_\alpha(z) \). The application of the criterion thus amounts to an inspection of the variations in sign of the coefficients of the characteristic polynomials of the \( B_\alpha \).

In some cases it is possible to choose the vector \( z \) so that each \( \cos \phi(p, q, \alpha) \) appearing in (2.5) is unity; then \( \lambda_{\alpha p} - S_\alpha(z) \) is a \( k \)-quadratic form in the \( s_q \).

**References**


University of British Columbia