LAPLACE SERIES AND SETS OF
LOGARITHMIC CAPACITY ZERO

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In the principal theorems of [2] and [3] sets of capacity zero ("capacity" will mean "logarithmic capacity" in this note) appear in the hypotheses. More specifically, conditions like finiteness of generalized Laplacians or of Poisson sums are assumed to hold everywhere except possibly on a closed set of capacity zero; these sets play the role of sets of uniqueness. The present note will show that the conclusions of these theorems become false if the sets of capacity zero are replaced by any larger class of closed sets.

We begin with the following known theorem [6, p. 84].

THEOREM A. *If e is a plane compact set of positive capacity, then there exists a positive measure μ, concentrated on e, such that the potential*

$$(1) \qquad v(z) = \int_e \log |\zeta - z| \, d\mu(\zeta)$$

is continuous in the whole plane.

By a measure we mean a finite real-valued (not necessarily nonnegative) completely additive set function defined for all Borel sets.

Since v is subharmonic in the plane and harmonic in the complement of e, it follows that Theorems I and II of [2] become false if exceptional sets of positive capacity are admitted.

As $z \to \infty$, $v(z)$ differs from $\mu(e) \cdot \log |z|$ by a bounded harmonic function. To construct a nonconstant function v of the form (1) which is also regular at infinity, we take two disjoint closed subsets e_1 and e_2 of e, both of positive capacity, and then choose positive measures μ_i concentrated on e_i ($i = 1, 2$) such that the associated potentials v_i are continuous in the plane, and such that $\mu_1(e_1) = \mu_2(e_2)$. If $\mu = \mu_1 - \mu_2$, then $\mu(e) = 0$, and

$$(2) \qquad v(z) = \int_e \log |\zeta - z| \, d\mu(\zeta) = v_1(z) - v_2(z)$$

has the desired properties.

We now turn to series of spherical surface harmonics $Y_n(P)$, where P denotes a point of the surface S of the unit sphere in Euclidean 3-space. The notation and nomenclature will be the same as in [3],

and for the sake of brevity we shall not repeat it all.

THEOREM B. *Let E be a closed set of positive capacity on S. There exists a series $\sum_0^\infty Y_n(P)$ with the following properties:*

(i) $\sum_1^\infty Y_n(P)$ *does not vanish identically, but for every P in $S-E$ the series is $(C, 1)$-summable to 0 and Riemann-summable to 0.*

(ii) $-\sum_1^\infty Y_n(P)/n(n+1)$ *is the Laplace series of a continuous function F on S.*

(iii) *If, in addition, E is such that every diameter of S has either both or none of its end points in E, then, for every P in $S-E$ and for every $k > 1/2$, $\sum_0^\infty Y_n(P)$ is (C, k)-summable to 0.*

Consequently closed sets Z of positive capacity cannot be admitted in the hypotheses of Theorems 2.6 and 2.7 of [3].

PROOF. Let N be a point in $S-E$, and project S stereographically on the plane through the center of S, with N as center of projection, taking E into a plane set e of positive capacity. If P and Q on S correspond to z and ζ in the plane and if $V(P) = v(z)/2\pi$, where $v(z)$ is given by (2), then V is continuous on S and harmonic in $S-E$; also, if PQ denotes the shorter great circle arc on S between P and Q, we have

$$(3) \quad K(P, Q) \equiv \log \left\{ \frac{\sin(NP/2)\sin(NQ/2)}{\sin(PQ/2)} \right\} = -\log |\zeta - z|,$$

as is easily verified by considering the singularities of the two functions, and their harmonic character.

Let λ be the measure (concentrated on E) obtained by transplanting μ to S by means of our projection. Then

$$(4) \quad V(P) = -\frac{1}{2\pi} \int_E K(P, Q) d\lambda(Q) \qquad (P \text{ on } S).$$

Recalling that $\lambda(E) = 0$, (3) shows that $V(P)$ differs from

$$(5) \quad F(P) = \frac{1}{2\pi} \int_E \log \sin(PQ/2) d\lambda(Q) \qquad (P \text{ on } S)$$

by a constant (we draw the reader's attention to the analogy between (5) and the function Ωf defined in [3, p. 293]). The function F has the following properties:

(6) F is continuous and not constant on S;

(7) F is harmonic in $S-E$;

(8) $\iint_S F(P) dP = 0$, since $\lambda(E) = 0$.

Now define $Y_n(P)$ $(n \geq 1)$ by the requirement that

$$- \sum_1^\infty Y_n(P)/n(n+1)$$

be the Laplace series of F; this is possible by (8) and implies

(9) $$- \frac{Y_n(P)}{n(n+1)} = \frac{2n+1}{4\pi} \iint_S F(Q)P_n(\cos PQ)dQ \quad (P \text{ on } S, n \geq 1)$$

where P_n is the Legendre polynomial of degree n. Substituting (5) into (9) and reversing the order of integration, we obtain (compare [3, Theorem 5.2])

(10) $$Y_n(P) = \frac{2n+1}{4\pi} \int_E P_n(\cos PQ)d\lambda(Q) \quad (n \geq 0).$$

In other words, $\sum_0^\infty Y_n(P)$ is the Laplace-Stieltjes series of the measure λ. Note that Y_0 is defined by (10), and that $Y_0 = 0$.

By (7), $\sum_0^\infty Y_n(P)$ is Riemann-summable to 0 in $S - E$.

Let $s_n^{(k)}(\theta)$ be the nth Cesàro mean of order k of the series

$$\sum_{n=0}^\infty (2n+1)P_n(\cos\theta).$$

For $1/2 < k \leq 1$ and $0 < \theta < \pi$, it is known that [1, p. 277]

(11) $$\left| s_n^{(k)}(\theta) \right| < A_k n^{1/2-k} \theta^{-1-k}(\sin\theta)^{-1/2}.$$

Also [1, p. 274]

(12) $$\left| s_n^{(1)}(\theta) \right| \leq 2 \quad (\pi/2 \leq \theta \leq \pi).$$

By (10), the (C, k)-means of $\sum_0^\infty Y_n(P)$ are equal to

(13) $$\sigma_n^{(k)}(P) = \frac{1}{4\pi} \int_E s_n^{(k)}(PQ)d\lambda(Q).$$

For P in $S - E$, let E^* be the set of all Q in E such that $PQ < \pi$. Since every single point has λ-measure zero, we may replace E by E^* in (13); taking $k = 1$, (11) and (12) show that $s_n^{(1)}(PQ) \to 0$ boundedly as $n \to \infty$, for Q in E^*. Consequently $\sigma_n^{(1)}(P) \to 0$, and part (i) of the theorem follows.

Part (iii) is proved similarly if we note that for some $\delta > 0$ we have $\delta \leq PQ \leq \pi - \delta$ for every Q in E, and apply (11).

We conclude with an analogue of a theorem of Salem and Zygmund [4, p. 25]; compare also [5, p. 584].

THEOREM C. *If E is a closed subset of S, the following statements are equivalent*:

(a) *The capacity of E is positive.*

(b) *There exists a measure λ, concentrated on E and not identically zero, whose Laplace-Stieltjes series is $\sum_0^\infty Y_n(P)$, such that*

$$- \sum_1^\infty Y_n(P)/n(n+1)$$

is the Laplace series of a continuous function.

(c) *Same as (b), with "continuous" replaced by "bounded."*

PROOF. We have proved above that (a) implies (b), and (b) obviously implies (c). If (c) holds and $Y_n(P)$ is given by (10) then the function F whose Laplace series is $-\sum_1^\infty Y_n(P)/n(n+1)$ is given by (5). Since F is bounded, harmonic, and not constant in $S-E$, the capacity of E cannot be zero. Hence (c) implies (a).

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