ON STIELTJES INTEGRATION

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Among the theorems concerning the Stieltjes integral there are two which are established for integrals in one-dimensional space, but not in spaces of more than one dimension. These are (I) if \( \int f dg \) exists, \( f \) and \( g \) have no common discontinuity; (II) if \( \int f dg \) exists, and \( g \) is of bounded variation and \( t \) is its total-variation function, then \( \int f dt \) exists. The method of proof for one dimension does not extend to higher dimensions. In this note extensions of these theorems to \( n \) dimensions are proved for the ordinary Stieltjes integral and for a modified form of it.\(^2\)

1. Definitions. Throughout this note we shall assume that \( f \) is real-valued and bounded on a set \( D \) in the space \( \mathbb{R}^n \), and that \( g \) is real-valued on \( \mathbb{R}^n \). For each interval \( I \subset \mathbb{R}^n \) we define \( \Delta g_I \) in the usual way, as the sum of \( 2^n \) terms each of which is \( \pm 1 \) times the value of \( g \) at a vertex of \( I \). If \( B \) is a closed interval contained in \( D \), an extended partition of \( B \) is a set \( P = \{ I_1, I_2, \ldots, I_k, x_1, x_2, \ldots, x_k \} \) in

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which the $I_j$ are nonoverlapping closed intervals whose union contains $B$, and for each $j$ the point $x_j$ is in $D \cap I_j$. $P$ is a \textit{restricted partition} of $B$ if it is an extended partition and $\bigcup I_j = B$. The \textit{mesh} of $P$ is the greatest of the diameters of the $I_j$ in $P$. For each extended partition $P$, define

$$S(P) = \sum_{j=1}^{k} f(x_j) \Delta \sigma(I_j \cap B).$$

If this approaches a limit as the mesh of $P$ approaches 0, the limit is the \textit{modified Stieltjes integral} of $f$ with respect to $\sigma$ over $B$, and we denote it simply by $\int_B f(x) d\sigma(x)$. If $S(P)$ has a limit as mesh $P \to 0$ subject to the condition that $P$ is a restricted partition, the limit is the (ordinary) Stieltjes integral of $f$ with respect to $\sigma$ over $B$. Whenever the modified integral exists so does the ordinary integral, and the two are equal (the reason for defining the modified integral is that it possesses some desirable properties which the ordinary integral lacks). But it is easy to see that if $f = (f(x) \mathbf{1} x \in D)$, the ordinary Stieltjes integral of $f$ with respect to $\sigma$ over $B$ is identical with the modified Stieltjes integral of its restriction $f_B = (f(x) \mathbf{1} x \in B)$ with respect to $\sigma$ over $B$.

\section{Two lemmas}

If $\int_B f(x) d\sigma(x)$ exists, to each $\epsilon > 0$ corresponds $\delta(\epsilon) > 0$ such that if mesh $P < \delta(\epsilon)$, then $|S(P) - \int_B f(x) d\sigma(x)| < \epsilon$. Let us define $O(I)$ to be the oscillation of $f$ on $I \cap D$ if this is nonempty, and to be 0 if $I \cap D$ is empty. Since in any finite set of nonoverlapping closed intervals of diameter $< \delta$ those which meet $D$ can be included in a partition of mesh $< \delta$, we readily establish the following lemma.

\textbf{Lemma 1.} If the (modified Stieltjes) integral $\int_B f(x) d\sigma(x)$ exists, and $(\delta(\epsilon) : \epsilon > 0)$ is defined as above, and $\epsilon > 0$, and $I_1, \ldots, I_k$ are nonoverlapping closed intervals of diameter $< \delta(\epsilon/2)$, then

$$\sum_{i=1}^{k} O(I_i) \Delta \sigma(I_i \cap B) < \epsilon.$$ 

In order to save verbosity, by a \textit{hyperplane} we shall always mean a set $\{x : x(i) = C\}$, where $i$ is one of the numbers 1, \ldots, $n$ and $C$ is real. We define a sequence $C_1, C_2, \ldots$ of integers recursively by the relations

$$C_0 = 2, C_n = 2 + C_0 + \binom{n}{1} C_1 + \binom{n}{2} C_2 + \cdots + \binom{n}{n-1} C_{n-1}$$

$$(n = 1, 2, 3, \ldots).$$

\textbf{Lemma 2.} Let $\epsilon$ be a positive number, and let $I$ be a closed interval such that for each closed interval $I' \subset I$, $O(I') \Delta \sigma(I' \cap B) < \epsilon$. Let $I$ be
subdivided by \( k \) \((k \leq n)\) mutually perpendicular hyperplanes into non-overlapping intervals \( I_1, I_2, \ldots, I_{2^k} \) whose intersection contains a point \( x_0 \). Then

\[
O(I) \left| \Delta_0(I_j \cap B) \right| < C_k \epsilon \quad (j = 1, 2, \ldots, 2^k).
\]

For brevity we write \( G(I_j) \) for \( \Delta_0(I_j \cap B) \).

Without loss of generality we may assume that \( x_0 \) is the origin and that the \( k \) cutting hyperplanes are those defined by the \( k \) equations \( x^{(i)} = 0 \) \((i = 1, \ldots, k)\). For each \( I_j \), let the "signature" \( \sigma(I_j) \) be the set of integers \( i \) in \( \{1, 2, \ldots, k\} \) for which the \( i \)th coordinate of the center of \( I_j \) is negative. If \( \sigma = \sigma(I_j) \), we can use \( I(\sigma) \) as another name for \( I_j \). The number of elements in \( \sigma \) will be denoted by \( |\sigma| \).

We now establish inductively

(*) \( 1 \leq j \leq 2^k \), and \( \sigma = \sigma(I_j) \), then

\[
O(I) \left| G(I_j) \right| < C_{|\sigma|} \epsilon.
\]

First suppose \( \sigma \) empty, so that \( \sigma(I_j) = I(\phi) \). Then \( O(I_j) \geq O(I)/2 \), so

\[
O(I) \left| G(I_j) \right| \leq 2O(I) \left| G(I_j) \right| < 2\epsilon = C_0 \epsilon.
\]

Next suppose statement (*) true if \( |\sigma(I_j)| < h \); we prove it true if \( |\sigma(I_j)| = h \). There are \( 2^h \) intervals \( I_m \) in the set \( \{I_1, \ldots, I_{2^k}\} \) with \( \sigma(I_m) \subseteq \sigma(I_j) \); for simplicity we may assume the notation chosen so that these are \( I_{\phi} = I_1, I_2, \ldots, I_{2^h} = I_j \). The union of these intervals is a closed interval \( I^* \), and

\[
G(I_j) = G(I^*) - \sum_{i=1}^{i=h} G(I_i).
\]

Also \( O(I^*) \geq O(I_j) \geq O(I)/2 \). Letting \( \sum' \) denote the sum over all proper subsets of \( \sigma(I_j) \), we have

\[
O(I) \left| G(I_j) \right| \leq O(I) \left| G(I^*) \right| + \sum_{i=1}^{i=h} O(I) \left| G(I_i) \right|
\]

\[
\leq 2O(I^*) \left| G(I^*) \right| + \sum' C_{|\sigma|} \epsilon
\]

\[
< 2\epsilon + \sum_{i=0}^{h-1} \binom{h}{i} C_i \epsilon
\]

\[
= C_h \epsilon.
\]
So (*) holds for $|\sigma(I_j)| = k$, and by induction holds for $0 \leq |\sigma(I_j)| \leq k$. Since $C_0 \leq C_1 \leq \cdots \leq C_k$, the lemma is established.

3. A theorem on discontinuities. The interval function $\Delta_x$ is continuous at a point $x$ if to each $\epsilon > 0$ corresponds $\delta > 0$ such that whenever $I$ is a closed interval of diameter less than $\delta$ and having $x \in I$, $|\Delta_x I| < \epsilon$.

**Theorem I.** If the modified Stieltjes integral $\int_B f(x) \, dg(x)$ exists, and $x_0$ is a point of $B$ at which $f = (f(x) : x \in D)$ is discontinuous, then the interval function $(\Delta_x (I \cap B)) : I$ an interval is continuous at $x_0$.

Let $\epsilon$ be the oscillation of $f$ at $x_0$, and let $\gamma$ be positive. By Lemma 1, there exists $\delta > 0$ such that for every closed interval $I^*$ of diameter less than $2\delta$, $O(I^*)|\Delta_x (I^* \cap B)| < \gamma \epsilon$. Let $I$ be a closed interval of diameter less than $\delta$ with $x_0 \in I$; let $V$ be the vertex of $I$ farthest from $x_0$, and $C$ the vertex of $I$ farthest from $V$. Define $I^*$ to be the closed interval with center $C$ and a vertex at $V$; its diameter is twice that of $I$, hence less than $2\delta$. Also, except in the trivial case of degenerate $I$, $x_0$ is interior to $I^*$, so $O(I^*) \geq \epsilon$. The $n$ hyperplanes through $C$ divide $I^*$ into $2^n$ nonoverlapping closed intervals having $C$ in common, and $I$ is one of these. By Lemma 2,

$$\epsilon |\Delta_x (I \cap B)| \leq O(I^*) |\Delta_x (I \cap B)| < C_n \gamma \epsilon,$$

so $|\Delta_x (I \cap B)| < C_n \gamma$. Since $\gamma$ is an arbitrary positive number this completes the proof.

**Corollary.** If this (ordinary) Stieltjes integral $\int_B f(x) \, dg(x)$ exists, there is no point of $B$ at which the interval-function $(\Delta_x (I \cap B)) : I$ an interval and the restriction of $f$ to $B$, $f_B = (f(x) : x \in B)$ are both discontinuous.

4. Interval-functions of bounded variation. If $\Delta_x$ is of bounded variation over $B$, there exists a function $t$ in $R^n$ such that for each closed interval $I \subseteq B$, $\Delta_t I$ is the total variation of $\Delta_x$ over $I$.

**Theorem II.** Let $f$ be a bounded function on a domain $D$ in $R^n$, $g$ a function on $R^n$ such that $\Delta_x$ is of bounded variation on a closed interval $B \subseteq D$, and $t$ a function on $R^n$ such that $\Delta_t I$ is the total variation of $\Delta_x$ over $I$ for each closed interval $I \subseteq B$. If the (modified) Stieltjes integral $\int_B f(x) \, dg(x)$ exists, so does $\int_B f(x) \, dt(x)$.

Write $G(I)$ for $\Delta_x (I \cap B)$ and $T(I)$ for $\Delta_t (I \cap B)$, and let $M$ be the upper bound for $|f(x)|$ on $D$. Let $(\delta(\epsilon) : \epsilon > 0)$ be defined as before Lemma 1, and let $\epsilon$ be a positive number. There exists a finite set of
nonoverlapping closed intervals $J_1, \ldots, J_q$ whose union is $B$ such that

$$\sum_{i=1}^{q} |G(J_i)| > T(B) - \epsilon.$$ 

Without loss of generality we may suppose that the $J_i$ are obtained by cutting $B$ by hyperplanes, and have diameter less than $\delta(\epsilon/2)$.

Let $\delta'$ be a positive number less than the least of the edges of the intervals $J_1, \ldots, J_q$. We now investigate an extended partition $P = \{ I_1, \ldots, I_m, x_1, \ldots, x_m \}$ of mesh less than $\delta'$. For each $I_h$, the nonempty intervals in the set $I_h \cap J_1, \ldots, I_h \cap J_q$ are obtained by cutting $I_h$ by hyperplanes, and have a common point $x_h'$. If we define

$$c_h = \sup \{ O(I') \mid G(I') : I' \subset I_h \},$$

by Lemma 1 we have $c_1 + \ldots + c_m \leq \epsilon$. By Lemma 2

$$O(I_h) \mid G(I_h \cap J_i) \mid \leq C_n c_h \quad (i = 1, \ldots, q),$$

and at most $2^n$ intervals $J_i$ have points in common with $I_h$, so

$$\sum_i O(I_h) \mid G(I_h \cap J_i) \mid \leq 2^n C_n c_h.$$ 

Then

$$\sum_h O(I_h) T(I_h) = \sum_{i,h} O(I_h) T(I_h \cap J_i) \leq \sum_h 2^n C_n c_h + \sum_{i,h} 2M \{ T(I_h \cap J_i) - \mid G(I_h \cap J_i) \mid \}$$

$$< (2^n C_n + 2M) \epsilon.$$ 

Again by Lemma 1,

$$\sum_i O(J_i) \mid G(J_i) \mid < \epsilon,$$

whence

$$\sum_i O(J_i) T(J_i) \leq \sum_i O(J_i) \mid G(J_i) \mid + 2M \sum_i \{ T(J_i) - \mid G(J_i) \mid \}$$

$$< (2M + 1) \epsilon.$$ 

Let $\xi_i$ be the center of $J_i$. Both $x_h$ and $x_h'$ are in $I_h$, so $|f(x_h) - f(x_h')| \leq O(I_h)$; and unless $I_h \cap J_i$ is empty both $\xi_i$ and $x_h'$ are in $J_i$, whence $|f(\xi_i) - f(x_h')| \leq O(J_i)$. Hence
\[
\left| \sum_{h} f(x_h) T(I_h) - \sum_{i} f(\xi_i) T(J_i) \right|
\leq \left| \sum_{h} [f(x_h) - f(x_h')] T(I_h) \right| + \left| \sum_{i,h} [f(x_h') - f(\xi_i)] T(I_h \cap J_i) \right|
\leq \sum_{h} O(I_h) T(I_h) + \sum_{i,h} O(J_i) T(I_h \cap J_i)
< (2^nC_n + 4M + 1)\varepsilon.
\]

If \( P^* = \{I_1^*, \ldots, I_n^*, x_1^*, \ldots, x_n^*\} \) is any extended partition also of mesh less than \( \delta' \), the same argument applies to it as to \( P \), so

\[
\left| \sum_{h=1}^{m} f(x_h) T(I_h) - \sum_{i=1}^{l} f(x_i^*) T(I_i^*) \right| < 2(2^n C_n + 4M + 1)\varepsilon.
\]

By the Cauchy criterion the limit of the sum \( \sum f(x_h) T(I_h) \) exists as mesh \( P \) tends to zero, and by definition this limit is \( \int_B f(x) dt(x) \).

**Corollary.** If \( \Delta_0 \) is of bounded variation on \( B \), and \( t \) is a function such that \( \Delta_0I \) is the total variation of \( \Delta_0 \) on \( I \) for each closed interval \( I \subseteq B \), and \( f \) is bounded and the (ordinary) Stieltjes integral \( \int_B f(x)dg(x) \) exists, so does \( \int_B f(x) dt(x) \).

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