A TAUBERIAN THEOREM FOR $\alpha$-CONVERGENCE OF CESÀRO MEANS

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The object of this note is to generalize certain Tauberian results proved by Gehring [3] for summability $(C, k; \alpha)$. The notation is as in [3], with the following additional definitions: If $k > -1$, then $A_n^k, B_n^k$ denote the $n$th Cesàro sums of order $k$ for the series $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, where $b_n = na_n$. $A_n^{-1}, B_n^{-1}$ denote $a_n, b_n$. Summability $(C, -1; \alpha)$ of $\sum a_n$ will be taken to mean summability $(C, 0; \alpha)$ of $\sum a_n$ together with the condition that sequence $\{na_n\}$ be $\alpha$-convergent to 0.

Gehring’s Tauberian theorems are:

**Theorem 4.3.2.** Suppose that $0 \leq \alpha \leq 1$ and that $\sum a_n$ is summable $(A; \alpha)$ to $S$. If the sequence $\{na_n\}$ is $\alpha$-convergent to 0, $\sum a_n$ is $\alpha$-convergent to $S$.

**Theorem 4.3.3.** Suppose that $0 \leq \alpha \leq 1$ and that $\sum a_n$ is summable $(A; \alpha)$ to $S$. Then $\sum a_n$ is $\alpha$-convergent to $S$ if and only if the sequence $\{(a_1 + \cdots + na_n)/n\}$ is $\alpha$-convergent to 0.

**Theorem 4.3.4.** Suppose that $0 \leq \alpha \leq 1$ and that $\sum a_n$ is $\alpha$-convergent. If the sequence $\{na_n\}$ is $\alpha$-convergent to 0, $\sum a_n$ is summable $(C, k; \alpha)$ to its sum for every $k > -1$.

These will be used in the proof of the following:

**Theorem.** Suppose that $0 \leq \alpha \leq 1$ and that $\sum a_n$ is summable $(A; \alpha)$ to $S$. Then, for $r \geq -1$, $\sum a_n$ is summable $(C, r; \alpha)$ to $S$ if and only if the sequence $\{B_n^r/C_n^{r+1}\}$ is $\alpha$-convergent to 0.

**Proof of Theorem.** Necessity. If $r = -1$ this follows immediately from the definition of summability $(C, -1; \alpha)$. If $r > -1$ then by the consistency theorem for $(C, r; \alpha)$ summability (Gehring [3, Theorem 4.2.1]) it follows that both sequences $\{S_n^r\}, \{S_n^{r+1}\}$ are $\alpha$-convergent to $S$. By Hardy [1, Equation (6.1.6)],

$$S_n^r = S_n^{r+1} + \frac{1}{r+1} \frac{B_n^r}{C_n^{r+1}},$$

and the result follows since a linear combination of sequences summable $(C, k; \alpha)$ is itself summable $(C, k; \alpha)$.

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Sufficiency. If \( r > -1 \) it may be shown\(^1\) as in Szász \([4, \S 1]\), that

\[
\frac{1}{y + 1} \sum_{n=0}^{\infty} S_n^{r+1} \left( 1 - \frac{1}{y + 1} \right)^n = \frac{r + 1}{y} \int_0^y \left( 1 - \frac{u}{y} \right)^r \phi(u) \, du,
\]

where \( \phi(u) = f(1 - 1/(u + 1)) \) and \( f(x) = \sum a_n x^n \).

Case (i). \( \alpha = 0, r > -1 \). Since \( \sum a_n \) is summable \( (A) \) to \( S \) it follows that \( \phi(u) \) tends to \( S \) as \( u \) increases. The right-hand side of (2), being the \( (r+1) \)th transform of \( \phi(u) \), also tends to \( S \); and so the sequence \( \{S_n^{r+1}\} \) is summable \( (A; \alpha) \) to \( S \).

Case (ii). \( 0 < \alpha \leq 1, r > -1 \). Putting

\[
\phi(u) = (r + 1) \int_0^1 (1 - v)^r (\varphi(vu)) \, dv,
\]

we get, from (2), that

\[
g(y) = \frac{1}{y + 1} \sum_{n=0}^{\infty} S_n^{r+1} \left( 1 - \frac{1}{y + 1} \right)^n,
\]

where \( \phi(u) \) now has bounded \( \alpha \)-variation over \( (0, \infty) \). Let

\[
V = \left[ \sum_{r=1}^{N} \left| g(y_r) - g(y_{r-1}) \right|^{1/\alpha} \right]^{\alpha}
= (r + 1) \left[ \sum_{r=1}^{N} \left( \int_0^1 (1 - v)^r \left[ \phi(vy_r) - \phi(vy_{r-1}) \right] \, dv \right)^{1/\alpha} \right]^{\alpha}.
\]

Then by Theorem 201 of \([5]\),\(^2\)

\[
V \leq (r + 1) \int_0^1 (1 - v)^r \left( \sum_{r=1}^{N} \left| \phi(vy_r) - \phi(vy_{r-1}) \right|^{1/\alpha} \right)^{\alpha} \, dv
\leq (r + 1) M \int_0^1 (1 - v)^r \, dv
= M,
\]

where \( M = V_{\alpha} \{ \phi(x); 0 \leq x < \infty \} \). Thus \( g(y) \) has bounded \( \alpha \)-variation over \( (0, \infty) \) and so the series \( \sum s_n \), where \( s_n = S_n^{r+1} - S_{n-1}^{r+1} \), is summable \( (A; \alpha) \) to \( S \). Further, by Hardy \([1, \text{Equation (6.1.6)}]\),

\[
ns_n = n(S_n^{r+1} - S_{n-1}^{r+1}) = \frac{B_n^r}{C_n^{r+1}},
\]

\(^1\) Note however the error in Szász's equation (2.4). There, and in the previous line, occurs an extraneous term \( (1 + 1/y)^{\alpha-1} \).

\(^2\) I am indebted to a referee for shortening my argument at this step.
so that sequence \( \{ns_n\} \) is \( \alpha \)-convergent to 0. Hence by Theorem 4.3.2 we have that \( \sum s_n \) and sequence \( \{S^{n+1}_n\} \) are \( \alpha \)-convergent to \( S \).

It is readily seen from Minkowski's inequality that the sum of two \( \alpha \)-convergent sequences is also \( \alpha \)-convergent, and we therefore deduce from (1) that \( \{S^n\} \) is \( \alpha \)-convergent to \( S \); i.e., \( \sum a_n \) is summable \( (C, r; \alpha) \) to \( S \).

Case (iii). \( r = -1 \). When \( \alpha = 0 \) the result reduces to Tauber's original theorem; when \( 0 < \alpha < 1 \) it follows from Theorem 4.3.2. For \( \alpha = 1 \) the result was proved by Hyslop [2, Theorem 4].

References