QUASI-EQUICONTINUOUS SETS OF FUNCTIONS¹

CHIEN WENJEN

The well-known theorem of analysis that if $F$ is a family of functions defined, equicontinuous, and uniformly bounded on a bounded closed set $E$ in $n$-dimensional real cartesian space $\mathbb{R}^n$, then from every sequence $\{f_n\}$ of functions of $F$ it is possible to select a uniformly convergent subsequence, has been recently generalized to various abstract spaces [1; 4; 6]. Consider a set $F$ of continuous functions on one topological space $X$ to another, $Y$. For any point $x$ of $X$ and any open set $W$ of $Y$ we denote by $(x, W)$ the totality of functions $f$ in $F$ for which $f(x) \in W$. The topology in $F$ obtained by employing all sets of the $(x, W)$ as a subbase in $F$ is called the $p$-topology by Arens [2]. The purpose of this note is to find the necessary and sufficient conditions that it be possible to select a subsequence converging pointwise to a continuous function from any given sequence of continuous functions and the necessary and sufficient conditions that a set of continuous functions be compact in the $p$-topology.

Definition. Let $\{f_n\}$ be a sequence of functions from a topological space $X$ to be metric space $Y$. $\{f_n\}$ is said to be $\varepsilon$-related at a point $x \in X$ if for every arbitrarily chosen $\varepsilon > 0$ there is a neighborhood $U(x)$ of $x$ such that, corresponding to each point $x' \in U(x)$, a positive number $N_\varepsilon(x, x')$ can be determined satisfying the condition:

$$\rho[f_n(x), f_n(x')] < \varepsilon$$

whenever $n > N_\varepsilon(x, x')$.

Definition. Let $F$ be a family of continuous functions from a topological space $X$ to a metric space $Y$. $F$ is called quasi-equicontinuous if in every infinite subset $Q$ of $F$ and at any point $x \in X$ there is a sequence $\{f_n\}$ contained in $Q$ which is $\varepsilon$-related at $x$.

Theorem. If $X$ is locally separable and $Y$ metric, a set of functions $F \subset Y^X$, where $Y^X$ denotes the set of all continuous functions from $X$ to $Y$, is compact under $p$-topology if and only if

- (1) $F$ is closed in $Y^X$,
- (2) $F(x) = \bigcup_{f \in F} f(x)$ is compact for every $x \in X$,
- (3) $F$ is quasi-equicontinuous.

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PROOF. I. Necessity. (1) $F$ is closed since $F$ is compact and $Y^X$ is a Hausdorff space under $\rho$-topology.

(2) Let $g_\varepsilon(f) = f(x)$. Then $g_\varepsilon$ is a continuous function of $f$ and the compactness of $F(x)$ follows from the compactness of $F$.

(3) Since compactness implies countable compactness, any infinite subset $Q$ of $F$ has a limit point $f$ contained in $F$. Let $\{x_n\}$ be an enumerable set contained and dense in a neighborhood $U(x_0)$ of a point $x_0$ in $X$. We can find a subset $\{f_n\}$ of $Q$ satisfying

$$
\rho[f_n(x_0), f(x_0)] < 1/n, \\
\rho[f_n(x_1), f(x_1)] < 1/n, \\
\cdots \\
\rho[f_n(x_n), f(x_n)] < 1/n, \\
$$

$n = 1, 2, 3, \cdots$

Then

$$
f_n(x_k) \to f(x_k), \quad k = 1, 2, 3, \cdots
$$
as $n \to \infty$.

Next we show that $f_n(x) \to f(x)$ at any point $x$ in $U(x_0)$. Suppose on the contrary that $f_n(x)$ does not converge to $f(x)$ at certain point $x'$ in $U(x_0)$. There exist an $\epsilon > 0$ and a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that

$$(A) \quad \rho[f_{n_i}(x'), f(x')] > \epsilon, \quad i = 1, 2, 3, \cdots$$

Let $g$ be a limit point of $\{f_{n_i}\}$ in $F$ and let a subsequence $\{f_{n_i}'\}$ of $\{f_{n_i}\}$ be so chosen that

$$
\rho[f_{n_i}'(x'), g(x')] < 1/i, \\
\rho[f_{n_i}'(x_1), g(x_1)] < 1/i, \\
\cdots \\
\rho[f_{n_i}'(x_i), g(x_i)] < 1/i, \\
$$

$i = 1, 2, 3, \cdots$

Then

$$
f_{n_i}'(x') \to g(x'), \\
f_{n_i}'(x_k) \to g(x_k), \quad k = 1, 2, 3, \cdots
$$
as $n_i'$ approaches to infinity. Now it is clear that

$\lim f_{n_i}'(x_k) = \lim f_{n_i}(x_k) = f(x_k) = g(x_k), \quad k = 1, 2, 3, \cdots$

We have therefore

$$f(x) = g(x)$$
for all $x$ in $U(x_0)$ on account of the continuity of the functions $f(x)$ and $g(x)$. It follows that

\[(B) \quad f_n(x') \rightarrow g(x') = f(x').\]

The contradiction between the relations (A) and (B) proves that $f_n(x)$ converges to $f(x)$ at any point $x$ in $U(x_0)$. In other words,

\[
\lim_{n \to \infty} \lim_{x \to x_0} f_n(x) = \lim_{x \to x_0} \lim_{n \to \infty} f_n(x).
\]

Hence $\{f_n\}$ is $\varepsilon$-related at $x_0$. The quasi-equicontinuity of the set of functions $F$ is proved.

II. Sufficiency. Since $F(x)$ is compact for any $x \in X$, the topological product $G = \prod_{x \in X} F(x)$ is compact. Consider the correspondence between $F$ and a subset $S$ of $G$ obtained by assigning to each $f \in F$ the point in $G$ with coordinates $f(x)$, $x$ ranging over $X$; this correspondence is a homeomorphism. In order to prove the compactness of $F$ it is sufficient to prove that $S$ is compact, that is, to prove that $S$ is closed in $G$.

Let $g$ be a limit point of $S$ with coordinates $g(x)$. There exists a sequence $\{f_n\} \subset F$ such that

\[
\rho[f_n(x_0), g(x_0)] < 1/n,
\]

\[
\rho[f_n(x_1), g(x_1)] < 1/n,
\]

\[
\cdots \cdots \cdots \cdots ,
\]

\[
\rho[f_n(x_n), g(x_n)] < 1/n, \quad n = 1, 2, 3, \ldots ,
\]

where $\{x_n\}$ is an enumerable set dense in a neighborhood $U(x_0)$ of $x_0$. By the quasi-equicontinuity of the set of functions $F$ there is a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that for each $\varepsilon > 0$ there is a neighborhood $V(x_0)$ of $x_0$ contained in $U(x_0)$ such that

\[
\rho[f_{n_i}(x_0), f_{n_i}(x)] < \varepsilon
\]

for any $x \in V(x_0)$ and all $n_i > N_{\varepsilon}(x_0, x)$. Now

\[
\rho[g(x_0), g(x_k)] < \rho[g(x_0), f_{n_i}(x_0)] + \rho[f_{n_i}(x_0), f_{n_i}(x_k)] + \rho[f_{n_i}(x_k), g(x_k)] < 3\varepsilon
\]

for any $x_k \in V(x_0)$, if $n_i$ is sufficiently large. By the same reasoning for any point $x$ in $V(x_0)$ there is a neighborhood $W(x)$ of $x$ contained in $U(x_0)$ such that

$^2$ That the $\varepsilon$-related condition was given by Hobson as a necessary and sufficient condition for interchange of order in repeated limits was pointed out by the referee [6, p. 409].
\[ \rho[g(x), g(x_k)] < 3\varepsilon \quad \text{if} \quad x_k \in \{ x_n \} \cap W(x). \]

Then
\[ \rho[g(x_0), g(x)] < \rho[g(x_0), g(x_j)] + \rho[g(x_j), g(x)] < 6\varepsilon \]
for any \( x \in V(x_0) \), where \( x_j \in V(x_0) \cap W(x) \). \( g(x) \) is therefore continuous at \( x_0 \), that is, \( g \) belongs to \( S \) and the closedness of \( S \) is proved.

**Corollary.** Let \( F \) be a family of continuous functions from a separable space \( X \) to a metric space \( Y \). The necessary and sufficient conditions that it be possible to select a subsequence converging pointwise to a continuous function from any given sequence of functions of \( F \) are:

1. \( F(x) \) is countably compact for any \( x \in X \),
2. \( F \) is quasi-equicontinuous.

**Corollary.** Let \((C)\) be the set of all continuous functions defined on the closed interval \((0, 1)\) and let \( F \) be any subset of \((C)\). \( F \) is weakly compact if and only if it is weakly closed and quasi-equicontinuous.

**References**


**University of California, Los Angeles**