THE NUMBER OF LATTICE POINTS IN AN 
$n$-DIMENSIONAL TETRAHEDRON$^1$

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Let the $a_i$ (1 ≤ $i$ ≤ $n$) be positive integers, no two of which have a common factor; let $A$ denote the product of the $a_i$.

In this paper we prove the following two exceedingly simple theorems, which we have not, to our surprise, seen before.

Theorem 1. If $\eta \equiv 0 \pmod{A}$, the number of solutions of

$$
\sum_{i=1}^{n} a_i x_i \leq \eta, \quad x_i \geq 0 \quad (1 \leq i \leq n)
$$

is a polynomial with rational coefficients in the quantities $\eta/A$ and the $a_i$.

Theorem 2. If $\eta \equiv 0 \pmod{A}$, the number of solutions of

$$
\sum_{i=1}^{n} a_i x_i = \eta, \quad \text{each } x_i \geq 0,
$$

is a polynomial with rational coefficients in the quantities $\eta/A$ and the $a_i$.

One should mention here that both theorems are completely trivial for $n = 1$. Theorem 2 is also trivial for $n = 2$, and indeed for the following reason. Clearly if $\eta \equiv 0 \pmod{a_1 a_2}$,

$$a_1 x_1 + a_2 x_2 = \eta$$

implies that $x_1 \equiv 0 \pmod{a_2}, \ x_2 \equiv 0 \pmod{a_1}$. Let $x_1 = a_2 x_1', \ x_2 = a_1 x_2'$. Hence we see that if $\eta \equiv 0 \pmod{a_1 a_2}$ the number of solutions of

$$a_1 x_1 + a_2 x_2 = \eta, \quad x_1 \geq 0, \ x_2 \geq 0$$

is the same as the number of solutions of

$$x_1 + x_1' = \frac{\eta}{a_1 a_2}, \quad x_1' \geq 0, \ x_2' \geq 0$$

which is clearly $(\eta/a_1 a_2) + 1$.

However, Theorem 2, while not deep, does not seem to be trivial for $n = 3$.

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$^2$ For the case when the $a_i$'s are all equal to 1, the result is well known.
EXAMPLE. For \( n = 3 \) (in Theorem 2) and \( \eta \equiv 0 \pmod{a_1a_2a_3} \), the number of solutions of \( \sum_{i=1}^{3} a_i x_i = \eta \) is

\[
\frac{\eta^2}{2a_1a_2a_3} + \frac{(a_1 + a_2 + a_3)}{2a_1a_2a_3} + 1.
\]

Theorems 1 and 2 are proved as follows. Following D. H. Lehmer [1] and D. C. Spencer [2] we write \( L(\eta) \) for the number of solutions of (1), when \( \eta \) is any non-negative real number; we use \( r(\eta) \) to denote the number of solutions of (2). Observe that \( r(\eta) = 0 \) if \( \eta \) is not an integer. We also write

\[
L^*(\eta) = L(\eta) - r(\eta)/2.
\]

We have (see [2])

\[
L^*(\eta) = \frac{1}{2\pi i} \int \frac{e^{s\nu} ds}{\prod_{i=1}^{n}(1 - e^{-a_i\nu})}
\]

where the integral is along the vertical line \( \sigma = c \) (\( c > 0 \); \( S = \sigma + it \), \( \sigma \) and \( t \) real). Now \( L^*(\eta) \) is a polynomial in \( \eta \) of degree \( n \), where the coefficients \( B_i \) \((1 \leq i \leq n)\) of \( \eta^i \), multiplied by \( A \), are polynomials in \( a_i \). The constant term \( B_0 \), which is a function of the \( a_i \) and \( \eta \), is periodic in \( \eta \), with a period equal to \( A \). (See G. H. Hardy [3] for the special case \( n = 2 \).)

From the periodic nature of \( B_0 \) it is trivial that when \( \eta \equiv 0 \pmod{A} \), \( B_0 = B_0(\eta; a_1, \ldots, a_n) = B_0(0; a_1, \ldots, a_n) \). The \( B_i \) are found as follows:

Our function \( f(S) \) has:

(i) A pole of order \( n+1 \) at \( S = 0 \). The residue \( P(\eta) \) here is a polynomial in \( \eta \) with coefficients which, when multiplied by \( A \), are polynomials (with rational coefficients) in the \( a_i \). This remark is true for all non-negative real \( \eta \).

(ii) A pole of order \( n \) at \( S = 2\pi i m \) \((m \text{ an integer} \neq 0)\). Here again, the residue \( Q(\eta) \) is a polynomial in \( \eta \) with coefficients which, when multiplied by \( A \), are polynomials (with rational coefficients) of the \( a_i \); provided \( \eta \) is an integer.

(iii) Simple poles at \( S = 2\pi i a_i \) where \( g \equiv 0 \pmod{a_i} \). Here again, the residue here is denoted by \( T_i(\eta) \), and has a period \( a_i \). (The argument in Hardy [3] generalizes at once to prove this statement.) \( T_i(\eta) \) is an infinite trigonometrical series. (Incidentally, in another note we show how to reduce these to finite trigonometrical sums.) Thus\(^8\)

\[
T_i(\eta) = \sum_{t \neq 0 \pmod{a_i}} \exp(\eta(2\pi i a_i)/2g) \cdot H_i = \prod_{j \neq i} (1 - \exp((-2\pi i a_i)/a_i)).
\]

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\(^8\) In fact for all \( \eta \geq 0 \), \( T_i(\eta) = \sum_{t \neq 0 \pmod{a_i}} \exp(\eta(2\pi i a_i)/2g) \cdot H_i \).
Finally we evaluate $r(\eta)$ when $\eta \equiv 0 \pmod{A}$ as follows. The latter condition on $\eta$ holds throughout the rest of the paper, and we shall not repeat it. Similar to our proof of (7) we easily establish that

\begin{equation}
L^*(\eta + 1/2) = P(\eta + 1/2) + R(\eta + 1/2) + \sum_{i=1}^{n} T_i(\eta + 1/2).
\end{equation}

The $R$ function here is not the same as the $Q$ introduced before. However, it remains, like $Q$, a polynomial in $\eta$ with coefficients which, when multiplied by $A$, are polynomials (with rational coefficients) in the $a_i$. Observe that

\begin{equation}
\sum_{i=1}^{n} T_i \left( \eta + \frac{1}{2} \right) = \sum_{i=1}^{n} T_i \left( \frac{1}{2} \right)
\end{equation}

\begin{equation}
= L^* \left( \frac{1}{2} \right) - P \left( \frac{1}{2} \right) - R \left( \frac{1}{2} \right)
\end{equation}

\begin{equation}
= 1 - P \left( \frac{1}{2} \right) - R \left( \frac{1}{2} \right)
\end{equation}

and

\begin{equation}
r(\eta) = 2 \{ L^*(\eta + 1/2) - L^*(\eta) \}.
\end{equation}

From the above equations we see that Theorems 1 and 2 follow. In fact $r(\eta)$ is a polynomial in $\eta$ of degree $n - 1$ with coefficients which, when multiplied by $A$, are polynomials (with rational coefficients) in the $a_i$. An identical remark is true about $L(\eta)$, except that we have, in this case, a polynomial of degree $n$ in $\eta$.

REFERENCES


University of Colorado and
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