ON A MIXED BOUNDARY VALUE PROBLEM OF HARMONIC FUNCTIONS

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The mixed boundary value problem has been treated recently by several authors [4; 5; 6; 7]. In this note we give an existence proof using subharmonic functions.

Consider a 2-dimensional multiply connected domain $D$ with the boundary $\Gamma$, which consists of $k$ closed curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$. The boundary curves $\Gamma_i$ ($i = 1, 2, \ldots, k$) are assumed to have continuous tangents.

A real-valued continuous function $f(\zeta)$ is defined on a part $\alpha$ of the boundary. (The boundary points are denoted by $\zeta$.) $\alpha$ consists of a finite number of disjoint closed curves or arcs $\alpha_i$. If $\alpha_i$ is an arc, then we denote by $P_i', P_i''$ its two end points. $|f(\zeta)| \leq M$.

The remaining part of $\Gamma$ we denote by $\beta$. Another real valued continuous function $g(\zeta)$ is defined on the closure of $\beta$.

The problem is to determine a function $u(z)$ in $D$, such that:

(a) $u(z)$ is harmonic in $D$,
(b) $\lim_{z \to \gamma} u(z) = f(\zeta)$ on $\alpha$, and $\lim \sup_{z \to \gamma} |u(z)| \leq M$ at $\zeta = P_i', P_i''$,
(c) $\lim_{z \to \gamma} u(z)$ is continuous for $\zeta$ on $\beta$,
(d) $\partial u/\partial n = g(\zeta)$ for $\zeta$ on $\beta$, where the differentiation is along the inward normal $n$.

We shall treat first the problem by assuming $g(\zeta) = 0$. Since Neumann's problem can be solved for smooth boundary, the assumption $g(\zeta) = 0$ on $\beta$ is not a restriction.

Define the class $\mathcal{F}$ of admissible functions $v(z)$ with the properties:

(a) $v(z)$ is continuous subharmonic in $D$,
(b) $\lim \sup_{z \to \gamma} v(z) \leq f(\zeta)$ for $\zeta$ on $\alpha$,
(c) $\lim_{z \to \gamma} v(z)$ exists and is continuous for $\zeta$ on $\beta$,
(d) $\lim \inf \Delta v/\Delta n \geq 0$ on $\beta$. The lim inf is taken along the normal pointing into the interior of $D$.

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2 The same problem was treated by R. Courant by the method of variational calculus [2, p. 40].
First, the class \( J \) is not empty, because all the constants \( \leq -M \) belong to it. Second, due to the property \( d \), the functions in \( J \) have \( M \) as an upper bound. To prove that, we make use of a theorem by E. Hopf [3], according to which any admissible subharmonic function \( v \) of class \( C'' \) does not attain its maximum on \( \beta \). If \( v \) is only continuous but not necessarily differentiable, Hopf's theorem remains valid for \( v \). For, Hopf's proof goes through also in this case if one replaces his condition \( L(v) \geq 0 \) by that of continuous subharmonic functions \( v \), and applies the maximum principle. Hence we have \( v \leq M \) in \( D + \beta \), for all \( v \) in \( J \).

The harmonic function \( u(z) \) will be determined by a modification of the well known method of O. Perron [9] on the Dirichlet problem.

We define the function \( u \) at each point \( z \) in \( D \) by

\[
(1) \quad u(z) = \text{l.u.b. } v(z),
\]

where the l.u.b. is taken over all \( v \) in \( J \). This definition is justified because the functions \( v \) are bounded from above.

**Lemma 1.** The function \( u(z) \) is harmonic in \( D \).

**Proof.** Suppose \( v_1, v_2 \in J \). We shall show that the function \( V(z) = \max (v_1, v_2) \) also belongs to \( J \). It is easily seen that the function \( V(z) \) has the properties \( a \) and \( b \). Consider a point \( P(\zeta) \) on \( \beta \). Since \( v_1 \) and \( v_2 \) are continuous at \( P \), we have \( \lim_{P \to \zeta} V(z) = \lim \max_{P \to \zeta} (v_1, v_2) = \lim_{P \to \zeta} [(v_1 + v_2) + |v_1 - v_2|] / 2 \), and hence \( V \) is continuous at \( P \). Let \( Q(z) \) be a point on the inward normal at \( P(\zeta) \), and let \( \Delta v_i = v_i(Q) - v_i(P), i = 1, 2 \). It is easily seen that \( \Delta V \leq \min (\Delta v_1, \Delta v_2) \). From this follows \( \lim \inf \Delta V / \Delta n \geq 0 \).

What is said for \( V \) is also valid for \( \max (v_1, v_2, \ldots, v_n) = V_n \). We may now carry out Perron's construction in the customary manner [1, p. 197] by forming a maximizing sequence \( \{ V_n \} \) of subharmonic functions in an arbitrary disk \( \Delta_i \), whose closure is contained in \( D \). The functions \( V_n \) which are equal to \( V_n \) outside and on the boundary of \( \Delta_i \), and equal to the Poisson integrals in \( \Delta_i \), form a nondecreasing sequence which converges to a harmonic function in \( \Delta_i \). This limit function is equal to \( u(z) \). Since \( \Delta_i \) is an arbitrary disk, the function \( u \) is harmonic in \( D \).

**Lemma 2.** The function \( u(z) \) determined by (1) satisfies \( \lim_{z \to \alpha} u(z) = f(\zeta_0) \) on \( \alpha \), except possibly the end points.

**Proof.** To prove the lemma we have to show that \( \lim \sup_{z \to \alpha} u(z) \leq f(\zeta_0) + \varepsilon \) and \( \lim \inf_{z \to \alpha} u(z) \geq f(\zeta_0) - \varepsilon \) for all \( \varepsilon > 0 \) and \( \zeta_0 \) on \( \alpha \). The
first inequality can be proved in the same manner as it is known for the Dirichlet problem [1, p. 198].

To prove the second inequality consider a simply connected subdomain $\Delta$ contained in $D$, such that a part $\delta$ of its boundary is on $\alpha$. The remaining part of the boundary of $\Delta$ we denote by $\gamma$. We can solve the Dirichlet problem in $\Delta$ with the boundary function $F = f(\xi) - \epsilon$, $\epsilon > 0$, on $\delta$ and $F = v_0$ on $\gamma$, where $v_0$ are the values on $\gamma$ of an admissible function $v$. We obtain a harmonic function $H(\xi)$ in $\Delta$.

Consider the function $W = H$ in $\Delta$ and $W = v$ in the rest of $D$. $W$ is subharmonic in $\Delta$, $\lim_{z \to z_0} W \leq f(\xi)$ on $\alpha$, $W = v$ on $\beta$, therefore $W$ is in $\mathcal{J}$.

Now $W = f(\xi) - \epsilon$ at $\xi_0$. As an admissible function $W(\xi) \leq u(\xi)$ and $\lim_{z \to z_0} u(\xi) \geq W(\xi_0) = f(\xi_0) - \epsilon$. Hence $\lim_{z \to z_0} u(\xi) = f(\xi)$ for all points of $\alpha$, except $P_1, P_1'$.

**Lemma 3.** $u(\xi)$ is continuous and $\partial u/\partial n = 0$ on $\beta$.

**Proof.** We shall make use of conformal mapping. Consider a simply connected subdomain $\Delta'$ contained in $D$. The boundary of $\Delta'$ is a simple closed curve. It consists of two arcs, one denoted by $\delta'$ is a closed subarc of $\beta$, while the other, denoted by $\gamma'$, is an open arc which lies in $D$. We can map $\Delta'$ conformally onto a semicircular domain $d$ so that $\gamma'$ goes into the circular arc and $\delta'$ into the bounding diameter. Denote this mapping by $S$.

Consider the maximizing sequence $\{V_n\}$, which was used for construction of $u$. Each $V_n$ has continuous boundary values $h_n$ on $\gamma'$. By $S$ the function $h_n$ goes into a continuous function $\phi_n$ on the circular arc of $d$. With the boundary function $\phi_n$ and its symmetric extension on the other semicircle we obtain a harmonic function $H_n$ in the closed disk. $H_n$ is symmetric with respect to the diameter and has a vanishing normal derivative on the diameter.

Now we transform $H_n$ back to $\Delta'$ by $S^{-1}$, and get a harmonic function $G_n$, which has a vanishing normal derivative on $\delta'$. $G_n = V_n$ on $\gamma'$. Since

$$\lim_{n \to \infty} \left( \frac{\Delta V_n}{\Delta n} - \frac{\Delta G_n}{\Delta n} \right) \geq \lim_{n \to \infty} \frac{\Delta V_n}{\Delta n} - \lim_{n \to \infty} \sup \frac{\partial G_n}{\partial n} \geq 0$$

on $\delta'$, we conclude that $G_n \geq V_n$ in $\Delta'$ (by making use again of Hopf's lemma [3]). The function equal to $G_n$ in $\Delta'$ and to $V_n$ outside of $\Delta'$ is an admissible function. Hence the functions $G_n$ form a maximizing sequence $\{G_n\}$, which converges to $u$. Hence $u$ is continuous and $\partial u/\partial n = 0$ on $\delta'$, consequently on $\beta$.

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\(^3\) This is the only step in the proof that cannot be used to treat the case of more than 2 independent variables.
Lemma 4. The function $|u(z)| \leq M$ in $D + \Gamma$.

Proof. From Lemmas 2 and 3 by the boundary behavior of $u$ it follows that $u \leq M$ in $D + \Gamma$. $u$ may not be defined at the end points $P'_1$, $P''_1$. Since $u \geq -M$ we have $|u| \leq M$.

Uniqueness of the solution. Suppose that there exists a harmonic function $u_1(z)$ which solves the problem and is different from $u(z)$. Consider the function $U = u - u_1$. $U$ is harmonic in $D$, $\lim_{z \to \infty} U = 0$ on the curves and open arcs $\alpha_i$, $U$ is continuous on $\beta$ and bounded at the points $P'_1$, $P''_1$ and furthermore $\partial U/\partial n = 0$ on $\beta$. We shall show that the maximum of $U$ is on $\alpha$. Suppose that the supremum of $U$ is $M_1$ and is equal to the supremum at a point $P$ which is one of the $P'_1$, $P''_1$. Consider a small circle $r$ around $P$, such that no other $P'_1$, $P''_1$ are in it. The part of $\beta$ inside the circle $r$ is denoted by $\beta_r$, the intersection of $D$ and $r$ by $d_r$. Let $m = \max U$ on $d_r$, $m < M_1$. By reflecting $d_r$ on $\beta_r$ (through a conformal mapping as in Lemma 3) we obtain a domain in which the harmonic function is bounded by the constant $m$ or zero, except at the point $P$. We can apply the extended maximum principle for harmonic functions [8, p. 115] to conclude that the supremum is not at $P$. The maximum is not on $\beta$ [3]. Therefore it is on $\alpha$ and $U \leq 0$ in $D + \Gamma$. By considering the function $-U$ instead of $U$, we get $-U \leq 0$. Both inequalities together imply $U = 0$ or $u = u_1$.

Bibliography


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