NONSEPARABILITY OF CERTAIN FINITE FACTORS

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Let $\mathcal{A}$ be a finite ring of operators with center Z; let Γ be the maximal ideal space of Z, and f the natural isomorphism from Z onto the continuous complex functions on Γ. Let tr denote the center-valued trace on $\mathcal{A}$. For each γ in Γ, we define an inner product $(\cdot, \cdot)_\gamma$ on $\mathcal{A}$ by $(A, B)_\gamma = f(\text{tr }(B^*A))(\gamma)$. The members A of $\mathcal{A}$ such that $(A, A)_\gamma = 0$ clearly form a two-sided ideal in $\mathcal{A}$, which we call $\mathcal{M}_\gamma$. Let $\mathcal{A}_\gamma$ be the quotient algebra $\mathcal{A}/\mathcal{M}_\gamma$, and ϕ_γ the natural map from $\mathcal{A}$ onto $\mathcal{A}_\gamma$. Since $\mathcal{M}_\gamma$ is closed under the * operation, a * operation is induced on $\mathcal{A}_\gamma$ as well. It is shown in [6] that $\mathcal{M}_\gamma$ is a maximal two-sided ideal, and that $\mathcal{A}_\gamma$ is a finite AW^* factor with numerical trace, the trace being given by $\tau(\phi_\gamma(A)) = f(\text{tr }(A))(\gamma)$. It is shown in [2] that any trace on a finite AW^* factor is automatically countably additive; therefore the results of [1] show that $\mathcal{A}_\gamma$ is weakly closed in its canonical representation: that is, its representation as left multiplication operators on the Hilbert space $\mathcal{H}_\gamma$ gotten by completing $\mathcal{A}_\gamma$ in the inner product $(\phi_\gamma(A), \phi_\gamma(B)) = \tau(\phi_\gamma(B)^*\phi_\gamma(A)) = (A, B)_\gamma$.

We shall show that the only time when $\mathcal{H}_\gamma$ is a separable Hilbert space is when it is trivially so: that the more common state of affairs is for $\mathcal{H}_\gamma$ to be nonseparable. From the nonseparability of $\mathcal{H}_\gamma$, furthermore, it follows that $\mathcal{A}_\gamma$ can have no representation whatsoever as operators on a separable Hilbert space.

LEMMA. *Let $\Gamma_1, \Gamma_2, \cdots$ be nonempty disjoint closed and open sets in Γ. Let $E_n = f^{-1}(\chi_{\Gamma_n})$, where χ_{Γ_n} is the characteristic function of Γ_n. Suppose $\mathcal{A}(E_n)$ has disjoint orthogonal equivalent projections $P_0^n, \cdots, P_{n-1}^n$ with $P_0^n + \cdots + P_{n-1}^n = E_n$. Let γ be in the closure of $\bigcup_{n=1}^\infty \Gamma_n$, but not in $\bigcup_{i=1}^\infty \Gamma_n$ itself. Then $\mathcal{H}_\gamma$ is nonseparable.*

PROOF. We shall exhibit a collection $\{A_{\bar\beta}\}$ of members of $\mathcal{A}$, where $\bar\beta$ ranges over the set of all sequences $\bar\beta = (\beta_0, \beta_1, \cdots)$ of zeros and ones, such that $\{\phi_\gamma(A_{\bar\beta})\}$ is an orthonormal set in $\mathcal{H}_\gamma$.

Define $\alpha_{h,k}^i$, where $0 \leqq i \leqq k$, $0 \leqq h \leqq 2^k - 1$, as follows: $\alpha_{h,k}^i = (-1)^{[h/2^i]}$, where $[r]$ denotes the largest integer $\leqq$ the real number r. For fixed k, and $i < j$, we have:

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$$\sum_{h=0}^{2^k-1} \overset{i}{\alpha}_{h,k}\overset{j}{\alpha}_{h,k} = \sum_{h=0}^{2k-1} (-1)^{[h/2^i]}(-1)^{[h/2^j]}$$
$$= \sum_{l=0}^{2^{k-j}-1} \sum_{h=l2^j}^{(l+1)2^j-1} (-1)^{[h/2^i]}(-1)^{[h/2^j]}$$
$$= \sum_{l=0}^{2k-j-1} (-1)^l \sum_{h=l2^j}^{(l+1)2^j-1} (-1)^{[h/2^i]}.$$

But $(-1)^{(h/2^i)}$ is positive and negative with equal frequency as h ranges from $l2^j$ to $(l+1)2^j-1$, so that $\sum_{h=0}^{2k-1} \overset{i}{\alpha}_{h,k}\overset{j}{\alpha}_{h,k}=0$.

Let $k(s)$ be the largest integer k such that $2^k \leqq s$. We now define A_n^i in $\mathcal{A}(E_n)$, for any positive integer n, and $2^i \leqq n$:

$$\overset{i}{A}_n = (n/2^{k(n)})^{1/2}\left(\sum_{h=0}^{2^{k(n)}-1} \overset{i}{\alpha}_{h,k(n)}\overset{n}{P}_h\right).$$

Then $\mathrm{tr}(A_n^{i*}A_n^j)=0$ if $i\neq j$, and $\mathrm{tr}(A_n^{i*}A_n^i)=E_n$. Furthermore, $\|A_n^i\| =(n/2^{k(n)})^{1/2}\leqq(2)^{1/2}$. Thus, given any sequence $\bar{\imath}=(i_1, i_2, \cdots)$ with $2^{i_n}\leqq n$, the partial sums $A_1^{i_1}+\cdots+A_n^{i_n}$ converge strongly to a member $A^{\bar{\imath}}$ of $\mathcal{A}$. If $\bar{\imath}$ and $\bar{\jmath}$ are two such sequences, and $i_n\neq j_n$ for all $n>n_0$, then $\mathrm{tr}(A_1^{i_1*}A_j^{j_1}+\cdots+A_n^{i_n*}A_n^{j_n})=\mathrm{tr}(A_1^{i_1*}A_1^{j_1}+\cdots+A_{n_0}^{i_{n_0}*}A_{n_0}^{i_{n_0}})$ for all $n>n_0$, so that, by continuity of the trace, $\mathrm{tr}(A^{\bar{\imath}*}A^{\bar{\jmath}}) =\mathrm{tr}(A_1^{i_1*}A_1^{j_1}+\cdots+A_{n_0}^{i_{n_0}*}A_{n_0}^{j_{n_0}})$, and therefore $f(\mathrm{tr}(A^{\bar{\imath}*}A^{\bar{\jmath}}))$ vanishes outside $\Gamma_1\cup\cdots\cup\Gamma_{n_0}$, and in particular, vanishes at γ; consequently, $\phi_\gamma(A^{\bar{\imath}})$ is orthogonal to $\phi_\gamma(A^{\bar{\jmath}})$. Furthermore, $A^{\bar{\imath}*}A^{\bar{\imath}}\geqq A_n^{i_n*}A_n^{i_n}$ for all n, so that $f(\mathrm{tr}(A^{\bar{\imath}*}A^{\bar{\imath}}))(\delta)\geqq f(\mathrm{tr}(A_n^{i_n*}A_n^{i_n}))(\delta)=1$ for $\delta\in\Gamma_n$; and finally, $1\geqq\mathrm{tr}\,(A_1^{i_1*}A_1^{i_1}+\cdots+A_n^{i_n*}A_n^{i_n})$ for all n, so that $1\geqq\mathrm{tr}\,(A^{\bar{\imath}*}A^{\bar{\imath}})$. Thus $f(\mathrm{tr}\,(A^{\bar{\imath}*}A^{\bar{\imath}}))(\delta)=1$ in $\bigcup_{n=1}^\infty \Gamma_n$, and therefore also in the closure of $\bigcup_{n=1}^\infty \Gamma_n$, and in particular for $\delta=\gamma$.

Now, if $n\geqq 4$ let $l(n)=k(k(n))-1$, and let $i_n(\bar{\beta})=2^{l(n)}\beta_0+2^{l(n)-1}\beta_1 +\cdots+2\beta_{l(n)-1}+\beta_{l(n)}$, for any dyadic sequence $\bar{\beta}=(\beta_0, \beta_1, \cdots)$; if $n=1, 2, 3$ let $i_n(\bar{\beta})=1$. If $\bar{\beta}\neq\bar{\alpha}$, then $i_n(\bar{\beta})$ can equal $i_n(\bar{\alpha})$ only for finitely many n. Further, $\bar{\imath}_n(\bar{\beta})\leqq 2\cdot 2^{l(n)}\leqq k(n)$, so $2^{i_n(\bar{\beta})}\leqq n$. Then if $\bar{\imath}(\bar{\beta})=(i_1(\bar{\beta}), i_2(\bar{\beta}), \cdots)$, we write $A_{\bar{\beta}}$ for $A^{\bar{\imath}(\bar{\beta})}$. The set $\{\phi_\gamma(A_{\bar{\beta}})\}$ is then our required orthonormal set in $\mathcal{H}_\gamma$.

If $\mathcal{B}$ is a finite factor with numerical trace τ, and $\mathcal{H}$ is the Hilbert space gotten by completing $\mathcal{B}$ in its τ-norm, then $\mathcal{B}$ has coupling constant[2] equal to 1 in its canonical representation $\mathcal{B}\to\lambda(\mathcal{B})$ on $\mathcal{H}$. Any 1-1 representation ψ of $\mathcal{B}$ as a ring of operators with coupling constant $\leqq 1$ can be constructed (up to unitary equivalence) by choosing an appropriate projection P in $\lambda(\mathcal{B})'$ and restricting $\psi(\mathcal{B})$ to

[2] For definition and properties of the coupling constant, see [4].

$P\mathcal{H}$; and if $\mathcal{H}$ is nonseparable, then since $\lambda(\mathcal{B})$ must be type II_1, and hence also $\lambda(\mathcal{B})'$, it follows that $P\mathcal{H}$ is nonseparable. On the other hand, any representation of $\mathcal{B}$ with coupling constant $\geqq 1$ can be constructed (again, up to unitary equivalence) by using a Hilbert space which has $\mathcal{H}$ as a subspace, and therefore is again nonseparable if $\mathcal{H}$ is. Thus, if $\mathcal{H}$ is nonseparable, $\mathcal{B}$ has *no* isomorphic representation as a ring of operators on a separable Hilbert space.

THEOREM. (a) *If $\{\gamma\}$ is open, then $\mathcal{H}_\gamma$ is separable if and only if $\mathcal{A}(E)$ can be isomorphically represented as a ring of operators on a separable Hilbert space, where E is $f^{-1}(\chi_{\{\gamma\}})$.* (b) *If $\{\gamma\}$ is not open, then $\mathcal{H}_\gamma$ is separable if and only if there is a closed and open set Γ_0 containing γ and such that $\mathcal{A}(E_0)$ is n-homogeneous for some positive integer n, where E_0 is $f^{-1}(\chi_{\Gamma_0})$.*

PROOF. (a) is evident from the discussion preceding this theorem and the fact that $\mathcal{H}_\gamma$ is precisely the Hilbert space of the canonical representation of the factor $\mathcal{A}(E)$.

If $\{\gamma\}$ is not open, and Γ_0 exists as in the condition in (b), then, by using the structure theory of [**3**], it is not difficult to see that $\mathcal{A}_\gamma$ is isomorphic to the algebra of $n\times n$ complex matrices, and therefore that $\mathcal{H}_\gamma$ has finite dimension n^2.

Suppose, finally, that $\{\gamma\}$ is not open, but that no such Γ_0 exists. Γ can be split into disjoint closed and open sets Γ_{I} and Γ_{II}, corresponding to the split-up of the identity of $\mathcal{A}$ into central projections E_{I} and E_{II} of type I and type II respectively.

(i) Suppose γ is in Γ_{I}. Then, by the structure theory of type I algebras, as described in [**3**], there are disjoint closed and open subsets $\Gamma_1, \Gamma_2, \cdots$ of Γ_{I} such that, denoting by E_i the projection $f^{-1}(\chi_{\Gamma_i})$, we have $\sum_{n=1}^{\infty} E_n = I$, and $\mathcal{A}(E_n)$ is n-homogeneous. By hypothesis, γ is not in any of the Γ_n; but γ is in the closure of $\bigcup_{n=1}^{\infty}\Gamma_n$, since this is all of Γ_{I}. Furthermore, since $\mathcal{A}(E_n)$ is n-homogeneous, it has n orthogonal equivalent projections $P_0^n, \cdots, P_{n-1}^n$ whose sum is E_n. Therefore the conditions of the preceding lemma are satisfied, and $\mathcal{H}_\gamma$ is nonseparable.

(ii) Suppose $\gamma\in\Gamma_{\mathrm{II}}$. Γ_{II} has some perfect measure μ, as shown in [**5**]. Since γ is not isolated, it is first category, and $\mu(\{\gamma\})=0$. Therefore there is a descending sequence $\Delta_1, \Delta_2, \cdots$ of closed and open sets containing γ, with $\mu(\Delta_n)<1/2^n$. Let $\Gamma_1=\Gamma_{\mathrm{II}}-\Delta_1$, and, inductively, $\Gamma_{n+1}=\Gamma_{\mathrm{II}}-(\Gamma_1\cup\cdots\cup\Gamma_n\cup\Delta_{n+1})$. Then the Γ_n are disjoint closed and open sets, and the complement of $\bigcup_{n=1}^{\infty}\Gamma_n$ is contained in $\bigcap_{n=1}^{\infty}\Delta_n$, hence has measure zero, so that $\bigcup_{n=1}^{\infty}\Gamma_n$ is dense in Γ_{II}. Furthermore, γ is not in any of the Γ_n. And finally, if $E_n=f^{-1}(\chi_{\Gamma_n})$, then $\mathcal{A}(E_n)$ is finite and type II, so that $\mathcal{A}(E_n)$ has n orthogonal, equivalent projections $P_0^n, \cdots, P_{n-1}^n$. Therefore again the hypotheses of our lemma are satisfied, and $\mathcal{K}_\gamma$ is inseparable.

This completes the proof.

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