that the rank of $H^i(K; J)$ is the sum of the ranks of $H^i(X; J)$ and $H^i(K, X; J)$ so equals

$$\binom{2n}{i} + 2^n \text{ if } i \text{ is even and } 0 < i < 2n,$$

and this completes the proof.

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A FIXED POINT THEOREM FOR CONTINUOUS MULTI-VALUED TRANSFORMATIONS

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1. Introduction. There are several different definitions of continuity for multi-valued transformations in existence in the literature [1]. Each definition is accompanied naturally by the question: which topological spaces $X$ have the property that, for each continuous multi-valued transformation $F$ of $X$ into $X$, there exists an $x \in X$ with $x \in F(x)$? This property is abbreviated F.p.p. and the point $x$ such that $x \in F(x)$ is called a fixed point under $F$. In [2], using one brand of continuity, Strother has shown that each closed and bounded interval $I$ of real numbers has the F.p.p., but that the square, $I \times I$, does not have it. Here, the concept of continuity will be the same as that in [2] and we shall answer the question above, restricting the topological spaces to be continuous curves (compact, locally connected, metric continua). More specifically, we shall prove that a continuous curve has the F.p.p. if and only if it is a dendrite [3, p. 88].

We shall employ the following characterization of continuity due to Strother [1]:

A multi-valued transformation $F$ of a space $X$ into a compact Hausdorff space $Y$ is continuous if and only if, for each $x_0 \in X$, it is true that:

1. $F(x_0)$ is closed,
2. $V$ open and containing $F(x_0)$ implies that there exists an open set $U'$ containing $x_0$ such that, if $x \in U'$, then $F(x) \subseteq V$, and

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(3) \( y_0 \in F(x_0), y_0 \in V, \) and \( V \) open imply that there is an open set \( U'' \) containing \( x_0 \) such that, if \( x \in U'' \), then \( F(x) \cdot V \neq \emptyset \).

2. Proofs. Each dendrite has a convex metric [4] and, since a dendrite \( D \) contains no simple closed curve, each convex metric for \( D \) is a convex metric with unique segments. We consider, in the first theorem, that the metric is convex, because the uniqueness of the segments simplifies some of the arguments.

**Theorem 1.** Each dendrite has the F.p.p.

**Proof.** Suppose the statement is false. Then there exists a dendrite \( D \) and a continuous multi-valued transformation \( F: D \rightarrow D \) without a fixed point. Let \( a \in D \) and consider the set \( E = \{ x \mid x \in D \text{ and there exists } z \in F(x) \text{ such that } x \text{ separates } a \text{ and } z \} \). We shall show that the initial supposition implies that \( E \) is nonempty and that \( E + \{ a \} \) is closed, which will provide a contradiction.

(i) \( E \) is nonempty. Let \( z \) be a point of \( F(a) \). Then \( z \neq a \), since \( a \notin F(a) \). Let \( D_z \) be the component of \( D - \{ a \} \) containing \( z \). Let \( U \) and \( V \) be open sets such that \( a \in U \), \( z \in V \) and \( U - V = \emptyset \). Let \( V' \) be open and connected and such that \( z \in V' \subset V \cdot D_z \). By the continuity of \( F \), there exists an open set \( U' \), about \( a \), such that \( x \in U' \) implies \( F(x) \cdot V' \neq \emptyset \). Let \( U'' = U' \cdot U \) and choose \( x \in U'' \cdot \text{seg (az)} \) such that \( x \neq a \). (We denote the unique segment from \( a \) to \( z \) by seg (az).) Let \( z' \in F(x) \cdot V' \). Then \( x \) separates \( z \) from \( a \), but does not separate \( z \) from \( z' \), since \( x \in V' \). Consequently, \( x \) separates \( z' \) from \( a \), which shows that \( E \) is not empty.

(ii) \( E + \{ a \} \) is closed. We call this set \( E' \). Suppose \( x \) is a limit point of \( E \), different from \( a \). Let \( \{ x_i \} \) be a sequence of points of \( E' \) such that \( x_i \neq a \), for each \( i \), and \( \{ x_i \} \rightarrow x \). If this is not an infinite sequence, then clearly \( x \in E' \). If \( \{ x_i \} \) is an infinite sequence, let \( z_i \) be a point, for each \( i \), such that \( z_i \in F(x_i) \) and such that \( x_i \) separates \( a \) from \( z_i \). If \( \{ z_i \} \) is infinite, let \( z \) be a limit point, select a subsequence converging to \( z \) and the corresponding subsequence of \( \{ x_i \} \). To save notation, suppose \( \{ z_i \} \) and \( \{ x_i \} \) denote the new sequences. If \( z \in F(x) \), then, from the characterization of continuity of \( F \), there exist open sets \( N, M, \) and \( V \) such that \( z \in N, x \in V, F(x) \subset M, N \cdot M = \emptyset, \) and \( x' \in V \) implies \( F(x') \subset M \). Then, if \( n \) is sufficiently large, \( x_n \in V \) and \( z_n \in N \), a contradiction, since \( x_n \in V \) implies \( z_n \in F(x_n) \subset M \). A similar argument provides a contradiction if \( \{ z_i \} \) is finite. Therefore \( z \in F(x) \).

We must also show that \( x \) separates \( z \) and \( a \). By choice, we have \( x \neq a \) and, since \( z \in F(x), z \neq x \). Also \( z \neq a \), for if it were, and \( \varepsilon = (1/3) \cdot \rho(a, x) \), then, for \( i \) sufficiently large, \( \rho(a, z_i) < \varepsilon \) and \( \rho(x, z) < \varepsilon \). Since
\(\rho(a, x) \leq \rho(a, x_i) + \rho(x_i, x)\), we would then have, for \(i\) sufficiently large, \(\rho(a, x_i) \geq 2\epsilon\) while \(\rho(a, z_i) < \epsilon\). Therefore, \(x_i\) would not separate \(a\) and \(z_i\), a contradiction. Hence, \(x \neq a \neq z \neq x\). Now, if \(x\) does not separate \(a\) and \(z\), then \(a, z\), and \(\text{seg} (az)\) belong to the same component \(D_x\) of \(D - \{x\}\). Let \(V\) be a connected neighborhood of \(z\) such that \(\overline{V} \subset D_x\). Let \(U\) be a neighborhood of \(x\) such that \(U \cdot (\overline{V} + \text{seg} (az)) = \emptyset\). For \(n\) sufficiently large, \(x_n \in U\) and \(z_n \in V\), and it is seen that \(x_n\) does not separate \(a\) from \(z_n\), a contradiction. Therefore, there exists a \(z \in F(x)\) such that \(x\) separates \(a\) and \(z\). Therefore, \(x \in E'\) and \(E'\) is closed.

(iii) The contradiction. Since \(E'\) is closed, there exists a point \(y \in E'\) such that \(\rho(a, y) = \sup \{\rho(a, x), x \in E'\}\). Since \(E\) is nonempty, \(y \neq a\); i.e., \(y \in E\). There exists, then, a point \(z \in F(y)\) such that \(y\) separates \(a\) and \(z\). Therefore, \(z\) belongs to a component \(D_y\) of \(D - \{y\}\) such that \(a \in D_y\). Let \(U\) and \(V\) be connected open sets such that \(y \in U\), \(z \in V\), \(U \cdot V = \emptyset\), and \(x \in U\) implies \(F(x) \cdot V = \emptyset\). Let \(\tilde{x} \in U \cdot (\text{seg} (yz) - \{y\})\) and \(\tilde{z} \in V \cdot F(\tilde{x})\). Then \(\tilde{x} \in D_y\), so \(y\) separates \(a\) from \(\tilde{x}\). Therefore, \(\rho(a, \tilde{x}) > \rho(a, y)\). Also \(\tilde{x}\) separates \(a\) and \(\tilde{z}\); for, if not, then the connected set \(K = V + \text{seg} (az) + \text{seg} (ay)\) contains \(y\) and \(z\), but not \(\tilde{x}\), contradicting the fact that \(\tilde{x}\) separates \(y\) and \(z\). Therefore, \(\tilde{x} \in E\). That \(\rho(a, \tilde{x}) > \rho(a, y)\) contradicts the fact that \(\rho(a, y) = \sup \{\rho(a, x), x \in E'\}\).

Hence the original assumption must be false; i.e., each dendrite \(D\) has the F.p.p.

We prove next that each nondegenerate continuous curve which is not a dendrite does not have the F.p.p., by demonstrating for such a space \(X\) a continuous multi-valued transformation of \(X\) into \(X\) which does not have a fixed point. The construction of the transformation is an obvious generalization of a construction in [2] and utilizes the fact that, under these conditions, the space must contain a simple closed curve.

**Theorem 2.** If \(X\) is a nondegenerate continuous curve which is not a dendrite, then \(X\) does not have the F.p.p.

**Proof.** \(X\) contains a simple closed curve \(C'\), if it satisfies the hypotheses. Let \(h\) be a homeomorphism of \(C'\) onto \(C\), the unit circle in the plane with center at the origin. Let the metric \(D\) on \(C\) be given by: \(D(x, y) = \text{length of the shortest arc containing } x \text{ and } y\). Also, since \(C'\) is an ANR, there exists an open set \(W\) containing \(C'\) and a continuous retraction \(r: W\) onto \(C'\). Let \(V\) be an \(\epsilon\)-neighborhood of \(C'\) such that \(V \subset W\). If \(x \in V\), let \(f(x) = (1/\epsilon)\rho(x, C')\). Then \(f\) is continuous and maps \(V\) into \([0, 1]\). We also define, for \(x \in V\), \(A(x)\) to be the arc of \(C\) with center at \(hr(x)\) and length \(2\pi f(x)\). (If \(f(x) = 0\), let
$A(x)$ be the point $hr(x)$.

Now define a multi-valued transformation $F': X \to X$ as follows:

$$F'(x) = \begin{cases} h^{-1}(A(x)), & \text{if } x \in V, \\ C', & \text{if } x \in X - V. \end{cases}$$

A straightforward argument will show that $F'$ is a continuous multi-valued transformation of $X$ onto $C'$. Let $R$ be a rotation of $C$ through $\pi$ radians. Define $F: X \to C'$ by $F = h^{-1}RhF'$. Now $F$ is a continuous multi-valued transformation, since $h^{-1}Rh$ is a homeomorphism of $C'$ onto $C'$. But $F$ does not have a fixed point. For, if $x \in X - C'$, then $F(x) \subseteq C'$, and, if $x \in C'$, then $F(x) = h^{-1}RhF'(x) = h^{-1}Rh(x)$, a point clearly not equal to $x$.

Therefore, $X$ does not have the F.p.p.

3. Conclusions. Putting Theorems 1 and 2 together, we have, therefore, proved:

**Theorem 3.** A nondegenerate continuous curve has the F.p.p. if and only if it is a dendrite.

If $X$ and $Y$ are continuous curves each containing more than one point, then $X \times Y$ contains a simple closed curve and, hence, is not a dendrite. We obtain thus the following statement, similar to a conclusion in [2]:

**Theorem 4.** If $X$ and $Y$ are nondegenerate continuous curves, then $X \times Y$ does not have the F.p.p.

**Bibliography**


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