that the rank of $H^i(K; J)$ is the sum of the ranks of $H^i(X; J)$ and $H^i(K, X; J)$ so equals

$$\binom{2n}{i} + 2^{2n} \text{ if } i \text{ is even and } 0 < i < 2n,$$

and this completes the proof.

UNIVERSITY OF CHICAGO

---

# A FIXED POINT THEOREM FOR CONTINUOUS MULTI-VALUED TRANSFORMATIONS

ROBERT L. PLUNKETT

1. **Introduction.** There are several different definitions of continuity for multi-valued transformations in existence in the literature [1]. Each definition is accompanied naturally by the question: which topological spaces $X$ have the property that, for each continuous multi-valued transformation $F$ of $X$ into $X$, there exists an $x \in X$ with $x \in F(x)$? This property is abbreviated F.p.p. and the point $x$ such that $x \in F(x)$ is called a fixed point under $F$. In [2], using one brand of continuity, Strother has shown that each closed and bounded interval $I$ of real numbers has the F.p.p., but that the square, $I \times I$, does not have it. Here, the concept of continuity will be the same as that in [2] and we shall answer the question above, restricting the topological spaces to be continuous curves (compact, locally connected, metric continua). More specifically, we shall prove that a continuous curve has the F.p.p. if and only if it is a dendrite [3, p. 88].

We shall employ the following characterization of continuity due to Strother [1]:

A multi-valued transformation $F$ of a space $X$ into a compact Hausdorff space $Y$ is continuous if and only if, for each $x_0 \in X$, it is true that:

(1) $F(x_0)$ is closed,

(2) $V$ open and containing $F(x_0)$ implies that there exists an open set $U'$ containing $x_0$ such that, if $x \in U'$, then $F(x) \subset V$, and

Presented to the Society, April 16, 1955; received by the editors March 5, 1955.

(3) $y_0 \in F(x_0)$, $y_0 \in V$, and $V$ open imply that there is an open set $U''$ containing $x_0$ such that, if $x \in U''$, then $F(x) \cdot V \neq \emptyset$.

2. **Proofs.** Each dendrite has a convex metric [4] and, since a dendrite $D$ contains no simple closed curve, each convex metric for $D$ is a convex metric with unique segments. We consider, in the first theorem, that the metric is convex, because the uniqueness of the segments simplifies some of the arguments.

THEOREM 1. *Each dendrite has the F.p.p.*

PROOF. Suppose the statement is false. Then there exists a dendrite $D$ and a continuous multi-valued transformation $F: D \to D$ without a fixed point. Let $a \in D$ and consider the set $E = \{x \mid x \in D$ and there exists $z \in F(x)$ such that $x$ separates $a$ and $z\}$. We shall show that the initial supposition implies that $E$ is nonempty and that $E + \{a\}$ is closed, which will provide a contradiction.

(i) *$E$ is nonempty.* Let $z$ be a point of $F(a)$. Then $z \neq a$, since $a \notin F(a)$. Let $D_z$ be the component of $D - \{a\}$ containing $z$. Let $U$ and $V$ be open sets such that $a \in U$, $z \in V$ and $U \cdot V = \emptyset$. Let $V'$ be open and connected and such that $z \in V' \subset V \cdot D_z$. By the continuity of $F$, there exists an open set $U'$, about $a$, such that $x \in U'$ implies $F(x) \cdot V' \neq \emptyset$. Let $U'' = U' \cdot U$ and choose $x \in U'' \cdot \text{seg}\,(az)$ such that $x \neq a$. (We denote the unique segment from $a$ to $z$ by seg $(az)$.) Let $z' \in F(x) \cdot V'$. Then $x$ separates $z$ from $a$, but does not separate $z$ from $z'$, since $x \notin V'$. Consequently, $x$ separates $z'$ from $a$, which shows that $E$ is not empty.

(ii) *$E + \{a\}$ is closed.* We call this set $E'$. Suppose $x$ is a limit point of $E$, different from $a$. Let $\{x_i\}$ be a sequence of points of $E'$ such that $x_i \neq a$, for each $i$, and $\{x_i\} \to x$. If this is not an infinite sequence, then clearly $x \in E'$. If $\{x_i\}$ is an infinite sequence, let $z_i$ be a point, for each $i$, such that $z_i \in F(x_i)$ and such that $x_i$ separates $a$ from $z_i$. If $\{z_i\}$ is infinite, let $z$ be a limit point, select a subsequence converging to $z$ and the corresponding subsequence of $\{x_i\}$. To save notation, suppose $\{z_i\}$ and $\{x_i\}$ denote the new sequences. If $z \notin F(x)$, then, from the characterization of continuity of $F$, there exist open sets $N$, $M$, and $V$ such that $z \in N$, $x \in V$, $F(x) \subset M$, $N \cdot M = \emptyset$, and $x' \in V$ implies $F(x') \subset M$. Then, if $n$ is sufficiently large, $x_n \in V$ and $z_n \in N$, a contradiction, since $x_n \in V$ implies $z_n \in F(x_n) \subset M$. A similar argument provides a contradiction if $\{z_i\}$ is finite. Therefore $z \in F(x)$.

We must also show that $x$ separates $z$ and $a$. By choice, we have $x \neq a$ and, since $z \in F(x)$, $z \neq x$. Also $z \neq a$, for if it were, and $\epsilon = (1/3) \cdot \rho(a, x)$, then, for $i$ sufficiently large, $\rho(a, z_i) < \epsilon$ and $\rho(x_i, x) < \epsilon$. Since

$\rho(a, x) \leqq \rho(a, x_i) + \rho(x_i, x)$, we would then have, for $i$ sufficiently large, $\rho(a, x_i) \geqq 2\epsilon$ while $\rho(a, z_i) < \epsilon$. Therefore, $x_i$ would not separate $a$ and $z_i$, a contradiction. Hence, $x \neq a \neq z \neq x$. Now, if $x$ does not separate $a$ and $z$, then $a$, $z$, and seg $(az)$ belong to the same component $D_x$ of $D - \{x\}$. Let $V$ be a connected neighborhood of $z$ such that $\overline{V} \subset D_x$. Let $U$ be a neighborhood of $x$ such that $U \cdot (\overline{V} + \text{seg } (az)) = \emptyset$. For $n$ sufficiently large, $x_n \in U$ and $z_n \in V$, and it is seen that $x_n$ does not separate $a$ from $z_n$, a contradiction. Therefore, there exists a $z \in F(x)$ such that $x$ separates $a$ and $z$. Therefore, $x \in E'$ and $E'$ is closed.

(iii) *The contradiction.* Since $E'$ is closed, there exists a point $y \in E'$ such that $\rho(a, y) = \sup \{\rho(a, x), x \in E'\}$. Since $E$ is nonempty, $y \neq a$; i.e., $y \in E$. There exists, then, a point $z \in F(y)$ such that $y$ separates $a$ and $z$. Therefore, $z$ belongs to a component $D_y$ of $D - \{y\}$ such that $a \notin D_y$. Let $U$ and $V$ be connected open sets such that $y \in U$, $z \in V$, $U \cdot V = \emptyset$, and $x \in U$ implies $F(x) \cdot V \neq \emptyset$. Let $\bar{x} \in U \cdot (\text{seg } (yz) - \{y\})$ and $\bar{z} \in V \cdot F(\bar{x})$. Then $\bar{x} \in D_y$, so $y$ separates $a$ from $\bar{x}$. Therefore, $\rho(a, \bar{x}) > \rho(a, y)$. Also $\bar{x}$ separates $a$ and $\bar{z}$; for, if not, then the connected set $K = V + \text{seg } (a\bar{z}) + \text{seg } (ay)$ contains $y$ and $z$, but not $\bar{x}$, contradicting the fact that $\bar{x}$ separates $y$ and $z$. Therefore, $\bar{x} \in E$. That $\rho(a, \bar{x}) > \rho(a, y)$ contradicts the fact that $\rho(a, y) = \sup \{\rho(a, x), x \in E'\}$.

Hence the original assumption must be false; i.e., each dendrite $D$ has the F.p.p.

We prove next that each nondegenerate continuous curve which is not a dendrite does not have the F.p.p., by demonstrating for such a space $X$ a continuous multi-valued transformation of $X$ into $X$ which does not have a fixed point. The construction of the transformation is an obvious generalization of a construction in [2] and utilizes the fact that, under these conditions, the space must contain a simple closed curve.

THEOREM 2. *If $X$ is a nondegenerate continuous curve which is not a dendrite, then $X$ does not have the F.p.p.*

PROOF. $X$ contains a simple closed curve $C'$, if it satisfies the hypotheses. Let $h$ be a homeomorphism of $C'$ onto $C$, the unit circle in the plane with center at the origin. Let the metric $D$ on $C$ be given by: $D(x, y) =$ length of the shortest arc containing $x$ and $y$. Also, since $C'$ is an ANR, there exists an open set $W$ containing $C'$ and a continuous retraction $r$: $W$ onto $C'$. Let $V$ be an $\epsilon$-neighborhood of $C'$ such that $V \subset W$. If $x \in V$, let $f(x) = (1/\epsilon)\rho(x, C')$. Then $f$ is continuous and maps $V$ into $[0, 1)$. We also define, for $x \in V$, $A(x)$ to be the arc of $C$ with center at $hr(x)$ and length $2\pi f(x)$. (If $f(x) = 0$, let

$A(x)$ be the point $hr(x)$.)

Now define a multi-valued transformation $F'$: $X \to X$ as follows:

$$F'(x) = \begin{cases} h^{-1}(A(x)), & \text{if } x \in V, \\ C' & \text{if } x \in X - V. \end{cases}$$

A straight-forward argument will show that $F'$ is a continuous multi-valued transformation of $X$ onto $C'$. Let $R$ be a rotation of $C$ through $\pi$ radians. Define $F$: $X \to C'$ by $F = h^{-1}RhF'$. Now $F$ is a continuous multi-valued transformation, since $h^{-1}Rh$ is a homeomorphism of $C'$ onto $C'$. But $F$ does not have a fixed point. For, if $x \in X - C'$, then $F(x) \subset C'$, and, if $x \in C'$, then $F(x) = h^{-1}RhF'(x) = h^{-1}Rh(x)$, a point clearly not equal to $x$.

Therefore, $X$ does not have the F.p.p.

3. **Conclusions.** Putting Theorems 1 and 2 together, we have, therefore, proved:

THEOREM 3. *A nondegenerate continuous curve has the F.p.p. if and only if it is a dendrite.*

If $X$ and $Y$ are continuous curves each containing more than one point, then $X \times Y$ contains a simple closed curve and, hence, is not a dendrite. We obtain thus the following statement, similar to a conclusion in [2]:

THEOREM 4. *If $X$ and $Y$ are nondegenerate continuous curves, then $X \times Y$ does not have the F.p.p.*

## BIBLIOGRAPHY

1. W. L. Strother, *Continuous multi-valued functions*, Dissertation, Tulane University, 1951.

2. ———, *On an open question concerning fixed points*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 988–993.

3. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, 1942.

4. R. H. Bing, *Partitioning continuous curves*, Bull. Amer. Math. Soc. vol. 58 (1952) pp. 536–556.

FLORIDA STATE UNIVERSITY