CHAIN HOMOTOPIE AND THE de RHAM THEORY

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Introduction. This note contains a method for constructing chain-homotopy operators suitable for the de Rham cohomology theory. In particular, it is proved that differentiably homotopic maps induce chain homotopic chain-mappings in the exterior algebra of differential forms (Formula 13 below; cf. pp. 80–81 of [1], where the same formula is obtained). This shows that the de Rham theory satisfies the "homotopy axiom" in the sense of S. Eilenberg and N. E. Steenrod (cf. [2]); hence the de Rham cohomology groups of a differentiably contractible manifold are trivial. This fundamental result is often referred to as the "Poincaré Lemma."

A simple generalization is given in the case of an almost product structure.

Almost complex and complex structures are investigated in §5; no genuine chain-homotopies are obtained, and in §6 is given an example which shows that $\bar{\partial}$-cohomology does not satisfy the homotopy axiom, even in the case of complex manifolds and analytic homotopies; this example is due to Professor K. Kodaira.

1. Definitions and notations. By "manifold" we mean "differentiable manifold of class $C^\infty," by "map," "map of class $C^\infty," etc.; and all notions such as tangent vector or differential form will be taken in their $C^\infty$-sense. Tangent vectors will always be taken to have been defined by the $C^\infty$-analogue of the definition given in §IV, Chap. II of [10].

If $U$ is a manifold, we denote by $T^1(U)$ the tangent bundle, by $T(U) = \bigoplus_{p=0}^\infty T^p(U)$ the bundle of exterior algebras of tangent vectors. Note that $T^0(U) = \mathbb{R}$ = the reals. By $\Phi(U) = \bigoplus_{p=0}^\infty \Phi^p(U)$ we denote the exterior algebra of differential forms; for our purposes, the most convenient definition is

\begin{equation}
\Phi^p(U) = \text{Hom}_{R(U)}[\times T^p(U), R(U)]
\end{equation}

where $R(U) = \Phi^0(U) = R$-module of $C^\infty$-maps $U \to R$, and $\times T^p(U)$ denotes the $R(U)$-module of cross-sections of $T^p(U)$. \{If $\Lambda$ is a commutative ring and $A, B$ are $\Lambda$-modules, $\text{Hom}_\Lambda(A, B)$ denotes the $\Lambda$-module of $\Lambda$-homomorphisms $A \to B$.\}

If $v, v' \subseteq \times T^p(U)$ are such that $v|_V = v'|_V$ ($v = v'$ "on $V"$) where $V$
is some open set of $U$, it is easy to see that $\phi v = \phi v'$ on $V$ for $\phi \in \Phi^p(U)$. Hence the definition of $\Phi^p(U)$ is a "local" one; and $\phi \in \Phi^p(U)$ can be given by giving its values on germs of cross-section; a germ of cross-section at $x \in U$ is the equivalence class of all cross-sections which agree (pairwise) in some neighborhood of $x$.

If $\phi \in \Phi^{p+q}(U)$ and $v \in \times T^p(U)$ we define the contraction $v \rhd \phi \in \Phi^{q}(U)$ by

$$(v \rhd \phi)v' = \phi(v \wedge v')$$

where $v' \in \times T^*(U)$.

The exterior derivative $d: \Phi^p \to \Phi^{p+1}$ is given by the formula

$$(d\phi)(v_1 \wedge \cdots \wedge v_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} v_i (\phi(v_1 \wedge \cdots \hat{v}_i \cdots \wedge v_{p+1}))$$

$$+ \sum_{i<j} (-1)^{i+j+1} [v_i, v_j] \wedge v_1 \wedge \cdots \hat{v}_i \cdots \hat{v}_j \wedge v_{p+1}$$

where the $v_i$ are germs of $\times T^1(U)$, $\phi \in \Phi^p(U)$, $[v_i, v_j] = v_i v_j - v_j v_i$ and $\cdots \hat{v}_i \cdots$ denotes the omission of the term with index $i$. The following will be useful:

**Lemma 1.** The homomorphism $d$ is uniquely characterized by:

(i) If $\phi \in \Phi^0(U), v \in \times T^1(U)$, $(d\phi)v = v \phi$,

(ii) If $\phi \in \Phi^0(U)$, $d^2 \phi = 0$,

(iii) if $\phi \in \Phi^p(U), \psi \in \Phi(U), d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^p \phi \wedge d\psi$.

Since locally $\Phi^1(U)$ is (isomorphic to) the Grassmann algebra generated by $\Phi^1(U)$ regarded as an $R(U)$-module, (ii) and (iii) imply (ii'):

$$d^2 = 0.$$  

If $U, V$ are manifolds, and $f: U \to V$ is a map, we denote by $f_*: T^1(U) \to T^1(V)$ the corresponding induced maps.

If $c$ is a differentiable (i.e., $C^\infty$) $p$-chain in $U$ and $\phi \in \Phi^p(U)$, we shall write $\phi \cdot c = f_* \phi$. Stokes's theorem then takes the form $(d\phi) \cdot c = \phi \cdot b c$, where $b$ denotes the boundary operator of the singular theory.

2. Almost product structure. We say that the manifold $U$ has almost product structure $(P, Q)$ if there are homomorphisms $P, Q: T^1(U) \to T^1(U)$ such that $T^1(U) = PT^1(U) \oplus QT^1(U)$ (direct sum). Thus for $v \in T^1(U)$, $v = Pv + Qv$ and hence if $v_i \in T^1(U)$ ($i = 1, \cdots , p$), then $v_1 \wedge \cdots \wedge v_p$ is a sum of terms each of which is the exterior

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2 More accurately: $\Phi(V)$ is the Grassmann algebra generated by $\Phi^1(V)$ for sufficiently small neighbourhoods $V \subset U$.

3 For fibre bundles the fibres of which are modules, a homomorphism is a fibre-preserving map which, restricted to any fibre, is a homomorphism in the algebraic sense.
product of \( r \) vectors of type \( P_{\nu} \), by \( s \) vectors of type \( Q_{\nu} \) where \( r+s=p \); for given \( r, s \) such a term is called a vector of "type \((r, s)\)"; and this process defines a unique projection operator

\[
\prod_{r,s}^*: T^{r+s}(U) \rightarrow T^{r+s}(U)
\]

onto the submodule of vectors of type \((r, s)\).

We then define projection operators \( \prod_{r,s}^*: \Phi^{r+s}(U) \rightarrow \Phi^{r+s}(U) \) by

\[
\prod_{r,s}^* \phi = \prod_{r,s}^* \phi
\]

If \( \phi = \prod_{r,s}^* \phi \) we say that \( \phi \) is of type \((r, s)\) (cf. [8]).

Let \( U, V \) be manifolds with almost product structures \((P, Q), (\overline{P}, \overline{Q})\) respectively. A map \( F: U \rightarrow V \) is said to be of type \((l, m)\) (in relation to these structures) if

\[
\prod_{l+m}^* F_* = F_* \prod_{r,s}^*.
\]

A map of type \((0, 0)\) is said to be admissible; the same definitions apply to any homomorphisms \( T(U) \rightarrow T(V) \) or \( \Phi(U) \rightarrow \Phi(U) \).

An examination of formula (2) shows that \( d = d_2 + d_1 + d_1' + d_2' \) where \( d_2, d_1, d_1', d_2' \) are of types \((2, -1), (1, 0), (0, 1), (-1, 2)\) respectively. \( d_2 = 0 \) leads to the following identities:

\[
d_2^2 = d_2 d_1' + d_1 d_2' = d_2 d_1'' + (d_1')^2 + d_1' d_2' \\
\]

\[
(3) \quad d_1^2 = d_1 d_1' + d_1 d_1'' + d_1' d_1 + d_1'' d_1 \\
= d_1' d_1'' + d_1' d_1'' + d_1' d_1 + d_1'' d_1 \\
= d_1' d_1'' + d_1' d_1'' = d_1' d_1' + (d_1')^2 + d_1' d_1' = (d_1')^2 = 0.
\]

In analogy to Lemma 1, we now define the \( R(U) \)-homomorphism \( d_P: \Phi^p(U) \rightarrow \Phi^{p+1}(U) \) by

(i) If \( \phi \in \Phi^p(U) \) and \( v \in T^1(U) \), then

\[
(d_P \phi)v = (Pv)\phi.
\]

(ii) If \( \phi \in \Phi^p(U) \), \( (d_P d + dd_P) \phi = 0 \).

(iii) If \( \phi \in \Phi^p(U) \) and \( \psi \in \Phi(U) \),

\[
d_P (\phi \wedge \psi) = d_P \phi \wedge \psi + (-1)^r \phi \wedge d_P \psi.
\]

It easily follows that

\[
(d_P d + dd_P) = 0.
\]

It is easily verified that \( 2d_2 + d_1 - d_2' \) satisfies these conditions; whence

\[
d_P = 2d_2 + d_1 - d_2'.
\]

Writing also

\[
d_Q = 2d_2' + d_1' - d_2',
\]
we see that \( d = d_P + d_Q \) and, by symmetry, that \( d_Q \) is related to \( Q \) as \( d_P \) is to \( P \).

Using (3), (4), we see that \( d_P^2 = d_1^2 + 2(d_1'd_1' + d_1''d_1') + d_1''^2 \). Hence, noting \( d|\Phi^0(U) = d_1' + d_1'' \) and appealing to Lemma 1, we have

**Lemma 2.** \( d_P^2 = 0 \) if and only if \( d = d_1' + d_1'' \); i.e., \( d_1' = d_1'' = 0 \); i.e., \( d_P = d_1' \), \( d_Q = d_1'' \).

It is not hard to prove that the conditions of Lemma 2 are equivalent to the "integrability" of the given almost product structure in which case we have a local product structure.

3. The / operation. Let \( U, V \) be manifolds. An obvious almost product structure is defined on \( U \times V \) by regarding \( P, Q \) as the (natural) projection operators associated with the direct sum decomposition \( T^i(U \times V) = T^i(U) \oplus T^i(V) \). We shall thus regard vector fields in \( U, V \) as lying, in an evident manner, in \( U \times V \). It is clear that the conditions of Lemma 2 pertain; we write \( d_U = d_P, d_V = d_Q \). \( d_U \) corresponds to "differentiation in \( U \) only."

Now, let \( \phi \in \Phi^{r+1}(U \times V) \) and let \( c \) be a singular \( r \)-chain in \( U \). Then (using a notation due to N. E. Steenrod, cf. [3]) we define \( \phi/c \in \Phi^r(V) \) by

\[
(\phi/c)(y) = (-1)^r [j^*_\nu(y \downarrow_\phi)] \cdot c
\]

where \( v \in T^*(V), y \in V \) and \( j_\nu: U \to U \times V \) is the map \( x \mapsto (x, y) \). Then, as is easily seen,

\[
(-1)^r d(\phi/c) = (-1)^{r+1} d_V \phi/c
\]

\[
= (-1)^{r+1} [d\phi/c - d_U \phi/c].
\]

Also, if \( v' \in T^{r+1}(V) \) we have

\[
(-1)^{r+1} (d_U \phi/c)(v')(y) = j^*_\nu(v' \downarrow_\phi) \cdot c
\]

\[
= (-1)^{r+1} [d_j^*_\psi(v' \downarrow_\phi)] \cdot c = (-1)^{r+1} j^*_\psi(v' \downarrow_\phi) \cdot bc
\]

Hence

\[
d\phi/c - \phi/bc = (-1)^r d(\phi/c)
\]

(cf. 2.9 in [3]).

Now assume that \( V \) has almost product structure \( (P, Q) \) and that \( U = U_P \times U_Q \). Define an almost product structure \( (\overline{P}, \overline{Q}) \) on \( U \times V = U_P \times U_Q \times V \) by

\[
\overline{P}T^i(U \times V) = T^i(U_P) \oplus PT^i(V),
\]

\[
\overline{Q}T^i(U \times V) = T^i(U_Q) \oplus QT^i(V).
\]
In this situation, formula (8) splits up into various components. We discuss one special case, namely that when \( c = c' \times x_0 \) where \( c' \) is an \( r \)-chain in \( U_F \) and \( x_0 \) is a point of \( U_Q \) regarded as a 0-chain. In this case the homomorphism \( \phi \rightarrow \phi/c \) is of type \((-r, 0)\) in relation to the almost product structures \((\overline{P}, \overline{Q})\), \((P, Q)\). By examining (8) in terms of its components we obtain:

\[
\begin{align*}
\delta_0 \phi/c &= (-1)^r \delta_0 \phi/c, \\
\delta_1 \phi/c &= (-1)^r \delta_1 \phi/c, \\
\delta_2 \phi/c &= (-1)^r \delta_2 \phi/c, \\
\delta_3 \phi/c &= (-1)^r \delta_3 \phi/c, \\
\delta_4 \phi/c &= (-1)^r \delta_4 \phi/c
\end{align*}
\]

(9)

from which, using (4) and (5), we obtain

\[
\begin{align*}
\delta_0 \phi/c - \phi/bc &= (-1)^r \delta_0 \phi/c, \\
\delta_0 \phi/c &= (-1)^r \delta_0 \phi/c.
\end{align*}
\]

(10)

4. Chain homotopies. Let us retain the notations of §3, let \( W \) be a third manifold and \( F: U \times V \rightarrow W \) a map. We define \( \lambda: \Phi(W) \rightarrow \Phi(V) \) by

\[
\lambda \psi = (-1)^{r+1}(F^* \psi)/c
\]

(11)

for \( \psi \in \Phi(W) \). Then, using (8) we get

\[
(d \lambda + (-1)^{r+1} \lambda d) \psi = (F^* \psi)/bc.
\]

(12)

Now, consider the case when \( c: I \rightarrow U \) is a singular 1-simplex and define \( f_t: V \rightarrow W \) by \( f_t(y) = F(c(t), y) \); then \( F \) represents a homotopy, and (12) becomes

\[
d \lambda + \lambda d = f_1^* - f_0^*
\]

(13)

showing that differentiably homotopic maps induce chain-homotopic homomorphisms.

Next, consider the almost product structures introduced in the second part of §3, and assume that \( F \) is admissible (in relation to these structures). The homomorphism \( \lambda \) defined by (11) in terms of an \( r \)-chain \( c \) “in \( U_F \)” will be denoted by \( \lambda_P \). Using (10), we get

\[
(d \lambda_P + (-1)^{r+1} \lambda_P d_F) \psi = F^* \psi/bc,
\]

(14)

\[
dQ \lambda_P + (-1)^{r+1} \lambda_P d_Q = 0
\]

and finally, in analogy to (13),

\[
d \lambda_P + \lambda_P d_F = f_1^* - f_0^*
\]

(15)

in other words: A homotopy consistent with a given almost product
structure induces chain-homotopies for the operator $d_p$; and similarly for $d_q$.

5. **Almost complex structure** (cf. [5; 6]). Let $M$ be an $m$-manifold, and let $CT(M) = T(M) \otimes_R C$ where $C$ are the complex numbers; and let $C(M) = C$-module of $C^\infty$-maps $M \to C$. We define

$$C^p(M) = \text{Hom}_{C(M)}[\times CT^p(M), C(M)]$$

and $C(M) = \sum_{p=0}^\infty C^p(M)$; cf. (1). We also define $d: C^p(M) \to C^{p+1}(M)$ by the formal analogue of (2); the definitions of $f_\ast, f^\ast$ are similarly extended. It is clear that the whole "complex" theory is analogous to the "real" theory; also, $C(M)$ is naturally isomorphic to $\mathcal{R}(M) \otimes_R C$, $C^p(M)$ to $\mathcal{R}^p(M) \otimes_R C$ and, under this isomorphism, $d$ corresponds to $d \otimes 1$.

We say that $M$ has a complex almost product structure if there are $C(M)$-homomorphisms $P, Q: CT^1(M) \to CT(M)$ such that $CT^1(M) = PCT^1(M) \oplus QCT^1(M)$, $P, Q$ being projections. It is clear that the theory of almost product structures (§2 above) has an exact analogue in this situation: and we take over, without change, the definitions of $I^s$, $\Pi^s_t$, "type $(r, s)$," $d = d_p + d_q$, complete with Lemma 2.

We say that $M$ has almost complex structure if it has complex almost product structure together with an isomorphism $k: CT^1(M) \to CT(M)$ such that $kPCT^1(M) = QCT^1(M)$, $kQCT^1(M) = PCT^1(M)$, $k^2 = 1$. Then $k$ can be extended to $k: C(M) \to C(M)$ (and with a slight abuse of notation!) $k: C(F)(M) \to C(M)$. We write $kv = \bar{v}$, $k\phi = \bar{\phi}$. In this case, in accordance with the usual notation, we write $\partial, \bar{\partial}$ for $d_p, d_q$. If $\bar{\partial}^2 = 0$ (cf. Lemma 2) the given almost complex structure is called integrable (cf. [5]).

It is well known that if $M$ has almost complex structure and $n$ complex dimensions, then it can be assigned a Hermitian metric (cf. [4, p. 209]) and in terms of this a duality operator $\ast: C^p(M) \to C^{2n-p}(M)$ and a scalar product $(\phi, \psi)$ for $\psi, \phi \in C^p(M)$; cf [1; 5; 7; 8]. These operations satisfy

$$(\prod^\ast_{r,s} \phi, \psi) = (\phi, \prod^\ast_{r,s} \psi),$$

and also, writing $\delta = -\ast \partial \ast$,

$$(\phi, \delta \psi) = (\bar{\partial} \phi, \psi)$$

if $\phi, \psi$ are forms with compact carriers (cf. [5]). We define
(17) \[ \Delta = 2(\partial \overline{\partial} + \overline{\partial} \partial). \]

Now, let \( U \) be a subdomain (i.e., an open set) of \( M \) such that the closure of \( U \) in \( M \) is compact. By \( \mathcal{L} \) denote the Hilbert space (in terms of the scalar product just introduced) of norm-finite differential forms on \( U \) and by \( \mathcal{J} \) the space of forms \( \phi \in C^\Phi(M) \) such that \( \overline{\partial} \phi = \partial \phi = 0 \) and \( \phi = 0 \) outside \( U \); then \( \mathcal{J} \) can be regarded as a subspace of \( \mathcal{L} \); we denote by \( F: \mathcal{L} \rightarrow \mathcal{J} \) the associated projection operator. There exists a "Green's operator" \( G: \mathcal{L} \rightarrow \mathcal{L} \) such that

(18) \[ \Delta G \phi = \phi - F \phi \]

(cf. [5]).

Define \( H, J: \mathcal{L} \rightarrow \mathcal{L} \) by

(19) \[ H = 2\partial (G - G \partial) + F, \]

\[ J = 2\partial G. \]

If the structure is complex, \( \partial \overline{\partial} = 0 \), \( \partial \overline{\partial} = 0 \) and hence \( \partial \Delta = \overline{\partial} \overline{\partial}, \partial \Delta = \partial \partial; \) hence in this case \( \Delta H = 0 \). If \( U \) is a closed, compact manifold, \( \partial G - G \partial = 0 \) and \( H = F \).

In the case of complex euclidean \( n \)-space, it is trivial that there exists a Green's operator \( G \) satisfying \( \Delta G \phi = \phi \) and, if \( \phi \) has a compact support, \( \partial G \phi = \overline{\partial} G \phi \). Hence, if we assume that \( U \) is an arbitrary subdomain of complex euclidean space, then \( H \phi = 0 \) provided that the support of \( \phi \) is compact relative to \( U \).

As is easily verified,

(20) \[ \partial J + J \partial = I - H. \]

Let \( V \) be another almost complex manifold; give to \( U \times V \) the natural induced almost complex structure; and by \( J_U, H_U, \partial_U \) denote the operators on \( C^\Phi(U \times V) \) associated with \( U \). Then, if \( \phi \) is some singular \( r \)-chain in \( U \), define \( L: C^\Phi(U \times V) \rightarrow C^\Phi(V) \) by

(21) \[ L \phi = J_U \phi / c. \]

It is easily seen that

(22) \[ (-1)^{r+1} \partial L + L \partial = L \partial_U \]

or, using (20),

(23) \[ ((-1)^{r+1} \partial L + L \partial) \phi = (I - H_U - \partial_U J_U) \phi / c. \]

Notice that, since there is no Stokes's formula in the geometrical sense for \( \partial \), no formula analogous to (8) can be obtained; similarly, no formulas analogous to (10) seem to exist, as singular chains cannot be closely related to almost complex structure.
In particular, if \( r = 0 \), \( \partial_U J_U / \epsilon = 0 \) and (23) becomes

\[
(- \bar{\partial} L + \partial \bar{\partial}) \phi = (I - H_U) \phi / \epsilon.
\]

Let \( W \) be a third almost complex manifold, \( F: U \times V \to W \) a map such that \( \partial F^* = F^* \bar{\partial} \), let \( c = u_1 - u_0 \) where \( u_0, u_1 \in U \), write \( f_j(v) = F(u_i, v) \), and \( \lambda = LF^*: C^j(W) \to C^j(V) \). Then, (24) gives

\[
(- \bar{\partial} L + \partial \bar{\partial}) \phi = (f_1^* - f_0^*) \phi - (H_U F^* \phi)_1 + (H_U F^* \phi)_0
\]

where \((H_U F^* \phi)_i = H_U F^* \phi | u_i \times V\).

If \( U \) is compact and connected, \( H_U = F_U \) where \( F_U \) is the projection onto the space of forms satisfying \( \partial_U \phi = \bar{\partial} \phi = 0 \), and \( \gamma^0_U \) (subspace of such forms of degree 0) is isomorphic to \( C \); then \((H_U F^* \phi)_i = (H_U F^* \phi)_0 \) and (25) becomes

\[
- \bar{\partial} \lambda + \lambda \partial = f_1^* - f_0^*.
\]

6. **An example** (cf. the Introduction). Let \( G \) be the multiplicative group consisting of matrices of the form

\[
z = \begin{bmatrix}
1 & z_1 & z_2 \\
0 & 1 & z_3 \\
0 & 0 & 1
\end{bmatrix}
\]

where \( z_i \in C \); let \( D \) be the (discrete) subgroup consisting of all \( z \) such that \( z_i \) are Gaussian integers. Then \( V = G / D \) (i.e., the space of right cosets \( z \cdot D \)) is a homogeneous compact complex manifold (which was first considered by Iwasawa). It is easily seen that (in classical notation) the holomorphic 1-forms

\[
w_1 = dz_1, \quad w_2 = dz_2 - z_3 dz_1, \quad w_3 = dz_3
\]

are right invariant on \( G \); they can hence be regarded as holomorphic 1-forms on \( V \). Further, it is not hard to verify that \( w_1, w_2, w_3 \) generate the \( \partial \)-homology group \( H^{1,0}_\partial(V) \) of forms of type \((1, 0)\). Hence

\[
\dim H^{1,0}_\partial(V) = 3.
\]

By the duality theorem of Kodaira-Serre (cf. [9; 11])

\[
\dim H^{2,3}_\partial(V) = 3.
\]

It is easy to verify that

\[
\psi_1 = w_2 \wedge w_3 \wedge \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3,
\]

\[
\psi_2 = w_3 \wedge w_1 \wedge \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3,
\]

\[
\psi_3 = w_1 \wedge w_2 \wedge \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3
\]
represent linearly independent elements of $H^{2,4}_G(V)$ and hence generate this group.

Now, every $t \in G$ induces the analytic homeomorphism $T_t: z \mapsto t \cdot z$ of $V$ onto itself; obviously each $T_t$ is homotopic to the identity. We have

\[
(T_t)^*w_1 = w_1, \quad (T_t)^*w_2 = w_2 - l_3 w_1 + l_1 w_3, \quad (T_t)^*w_3 = w_3
\]

and hence

\[
(T_t)^*\psi_1 = \psi_1 + l_3 \psi_2, \quad (T_t)^*\psi_2 = \psi_2, \quad (T_t)^*\psi_3 = \psi_3 - l_1 \psi_2
\]

showing that there is no chain-homotopy.

References


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