A THEOREM OF ÉLIE CARTAN

G. A. HUNT

André Weil [1] and Hopf and Samelson [2] have given a topological proof of the following theorem of Élie Cartan.

Two maximal Abelian subgroups of a compact connected Lie group $G$ are conjugate within $G$.

I present a simple metric proof.

**Lemma.** If $x$ and $y$ are elements of the Lie algebra $g$ of $G$ then $[x, A_x y]$ vanishes for some inner automorphism $A_x$ of $G$.

**Proof.** Because $G$ is compact one can define on $g$ a nonsingular bilinear form $(u, v)$ which is invariant: $([u, v], w) + (v, [u, w]) = 0$. We choose $\varepsilon$ in $G$ so that $(x, A_x y)$ attains its minimum for $\sigma = \varepsilon$; without loss of generality we may assume $\varepsilon$ to be the neutral element of $G$, and then $A_x y = y$. If now $z$ is any element of $g$ the function $(x, A_{\exp(\varepsilon t)} y)$ has a minimum for $t = 0$, so that its derivative vanishes there. Thus, keeping in mind that

$$\frac{d}{dt} A_{\exp(\varepsilon t)} y \bigg|_{t=0} = [z, y],$$

we have $(x, [z, y]) = 0$. From this equation and from the invariance of the bilinear form it follows that $([x, y], z) = 0$ for all $z$; this can happen only if $[x, y]$ vanishes, for the bilinear form is nondegenerate.

Before proving Cartan's theorem I recall some well-known facts: A maximal Abelian subgroup $\mathcal{C}$ of $G$ is a torus group; there is an element $x$ in the Lie algebra $\mathfrak{c}$ of $\mathcal{C}$ such that the one parameter group $\exp tx$ is dense in $\mathcal{C}$; if $y$ belongs to $g$ and $[x, y] = 0$, then $y$ must lie in $\mathfrak{c}$.

Matters being so, let $\mathcal{C}'$ be a second maximal Abelian subgroup of $G$ and $x'$ an element of its Lie algebra bearing the same relation with $y$ as $x$.
to \( \mathfrak{K}' \) as \( x \) does to \( \mathfrak{K} \). Now choose \( \sigma \) in \( \mathcal{G} \) so that \([x, A_x']\) vanishes. Then \( A_x' \) lies in \( \mathfrak{h} \); consequently \( A_x'(\exp tx') = \exp (tA_x') \) lies in \( \mathfrak{K} \) for every \( t \). So \( \mathfrak{K} \), being closed, includes the closure \( A_x'(\mathfrak{K}') \) of the one-parameter group \( A_x(\exp tx') \). Finally \( A_x(\mathfrak{K}') = \mathfrak{K} \), because both are maximal Abelian subgroups of \( \mathcal{G} \).

Since every element of \( \mathcal{G} \) can be written as \( \exp y \), the argument shows that every element of \( \mathcal{G} \) can be moved into \( \mathfrak{K} \) by an inner automorphism of \( \mathcal{G} \).

The referee has pointed out that the argument of the lemma above is very like one used by R. Bott [3] in another context.

**Bibliography**


**Cornell University**