UNIVALENCE OF CONTINUED FRACTIONS AND STIELTJES TRANSFORMS[1]

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1. **Introduction.** In this paper we establish a connection between the theory of univalent functions on the one hand and Stieltjes transforms and certain classes of continued fractions on the other. The only known result of approximately similar character is one, due to G. Szegö, which is mentioned in Corollary 2.1.

In §2 are some theorems on univalence of functions represented by Stieltjes transforms; these theorems depend upon the well known fact that (cf. [2][2]) a function $f(z)$ analytic in a convex region R is univalent in R if there exists a real constant α for which $e^{i\alpha}f'(z)$ is nowhere real in R. §3 contains similar results for functions represented by classes of continued fractions which may also be represented as a Stieltjes transform of §2. The domains of univalence obtained in these two sections are shown to have a maximal property.

The results of §§4 and 5 relate respectively to univalence and the star-like character of functions having certain continued fraction representations but not necessarily having representations as Stieltjes transforms. The domains of these two sections, which are obtained with the aid of some value region theorems for continued fractions, do not possess the maximal property of §§2 and 3.

It will be obvious that all of the subsequent theorems may be stated in an alternate form in terms of the reciprocal of the complex variable used in the given statement. For brevity such alternate statements have been omitted; we have, however, consistently used the complex variables x and z to correspond to expansions about the origin and infinity respectively.

2. **Theorems on Stieltjes transforms.** Following Theorem 3.1 we give an example of a function which is representable in the form $\int_0^1 d\phi(t)/(1+xt)$, where $\phi(t)$ is nondecreasing and bounded, which is not univalent in any larger region including Re $x \geqq -1$, $x \neq -1$. That all functions of this class are univalent in this region is shown by the following theorem.

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[1] The results of this paper are extracted from a Northwestern University thesis prepared under the direction of Professor W. T. Scott.

[2] Numbers in brackets refer to the bibliography at the end of this paper.

THEOREM 2.1. *If $\phi(t)$ is nondecreasing and bounded,* $0 \leqq t \leqq 1$, *then unless $\phi(t)$ increases only at $t=0$ the function $F(x)$ given by*

$$F(x) = \int_0^1 \frac{d\phi(t)}{1 + xt} \tag{2.1}$$

is univalent for $\mathrm{Re}\, x \geqq -1$, $x \neq -1$. *Moreover there exist functions of the form* (2.1) *not univalent in any larger region including* $\mathrm{Re}\, x \geqq -1$, $x \neq -1$.

For $\mathrm{Im}\, x \neq 0$, $\mathrm{Im}\, x$ and $\mathrm{Im}\, F(x)$ are of opposite sign. Hence univalence in each of the upper and lower half planes $\mathrm{Im}\, x > 0$ and $\mathrm{Im}\, x < 0$ assures univalence in their logical sum. Writing $x = u + iv$ we have

$$\mathrm{Im}\, F'(x) = 2\int_0^1 \frac{vt^2(1 + ut)}{|1 + xt|^2}\, d\phi(t).$$

We see that $\mathrm{Im}\, F'(x)$ is not zero in each of the regions $\mathrm{Re}\, x \geqq -1$, $\mathrm{Im}\, x > 0$ and $\mathrm{Re}\, x \geqq -1$, $\mathrm{Im}\, x < 0$, unless $t = 0$ is the only point of increase of $\phi(t)$. For any two real points x_1 and x_2 with $\mathrm{Re}\, x_1 > -1$, $\mathrm{Re}\, x_2 > -1$, the integral form of the difference $F(x_1) - F(x_2)$ shows that this difference is nonzero.

Complementing a result of Szegö who showed [7] that monotoneity of the third order for the sequence $\{c_i\}$ is a sufficient condition for the function $f(x)$ of Corollary 2.1 to be univalent in $|x| < 1$, we have this corollary:

COROLLARY 2.1. *A sufficient condition for the analytic function*

$$f(x) = c_0 - c_1x + c_2x^2 - \cdots, \qquad c_i \text{ real}, \qquad i = 0, 1, 2, \cdots,$$

to be univalent in $\mathrm{Re}\, x \geqq -1$, $x \neq -1$ *is that the sequence* $\{c_i\}$ *be totally monotone in the sense that* $\Delta^m c_n \geqq 0$, $m, n = 0, 1, \cdots$ *where* $\Delta^m c_n = c_n - C_{m,1}c_{n+1} + C_{m,2}c_{n+2} - \cdots + (-1)^n C_{m,n}c_{m+n}$.

The corollary follows when we apply a theorem of Hausdorff [3] that total monotoneity of the sequence $\{c_i\}$ is a sufficient condition for $f(x)$ to be representable in the form (2.1).

The example of Theorem 3.1 also shows the maximal property for the class of functions in the following theorem:

THEOREM 2.2. *If $\phi(t)$ is nondecreasing and bounded* $0 \leqq t \leqq 1$, *then unless $\phi(t)$ increases only at $t=0$ the function $F(x)$ given by*

$$F(x) = \int_0^1 \frac{x}{1 + xt}\, d\phi(t) \tag{2.2}$$

is univalent for Re $x \geqq -1$, $x \neq -1$. *Moreover there exist functions of the form* (2.2) *not univalent in any larger region containing this region.*

If we note that, for Im $x \neq 0$, we have Im x and Im $F(x)$ of the same sign, then considerations similar to those of Theorem 2.1 give the result. We defer the proof of the last sentence of the theorem until Theorem 3.1.

In connection with Theorem 3.3 we give a continued fraction which shows the "bestness" of the region of univalence in the following theorem:

THEOREM 2.3. *If $\phi(t)$ is nondecreasing and bounded for $-1 \leqq t \leqq 1$, then $F(z)$ given by*

$$F(z) = \int_{-1}^{+1} \frac{d\phi(t)}{z - t} \tag{2.3}$$

which is defined and analytic for z exterior to the cut $-1 \leqq \xi \leqq 1$, $z = \xi + i\eta$, is univalent for $|z| > 1$. Moreover there exist functions of the form (2.3) *not univalent in any larger neighborhood of infinity including $|z| > 1$.*

We employ an indirect proof supposing that $F(z_1) = F(z_2)$ for $z_1 \neq z_2$, $|z_1| > 1$, $|z_2| > 1$. Since for Im $z \neq 0$, Im z and Im $F(z)$ are of opposite sign we must have Im $z_1 \cdot$ Im $z_2 > 0$ or Im $z_1 = 0 =$ Im z_2. Then if we write $z = \xi + i\eta$, $F(z_1) = F(z_2)$, $z_1 \neq z_2$, yields

$$\begin{aligned} 0 = \int_{-1}^{+1} & \frac{t^2 - t(\xi_1 + \xi_2) + \xi_1\xi_2 - \eta_1\eta_2}{|z_1 - t|^2 |z_2 - t|^2} d\phi(t) \\ & + i \int_{-1}^{+1} \frac{t(\eta_1 + \eta_2) - \xi_1\eta_2 - \xi_2\eta_1}{|z_1 - t|^2 |z_2 - t|^2} d\phi(t) = I_1 + iI_2. \end{aligned}$$

Now both I_1 and I_2 must be zero so that we must have

$$0 = (\eta_1 + \eta_2)I_1 - (\xi_1 + \xi_2)I_2.$$

Computation of this combination of I_1 and I_2 yields an integral with the numerator of the integrand negative for $\eta_1 > 0$, $\eta_2 > 0$, $|z_1| > 1$, $|z_2| > 1$, $-1 \leqq t \leqq 1$. Hence we have a contradiction since the integral is negative.

A similar argument yields a contradiction for $\eta_1 < 0$, $\eta_2 < 0$. Hence we have univalence in each of the regions $|z| > 1$, Im $z > 0$ and $|z| > 1$, Im $z < 0$ and from previous remarks in their logical sum.

If we note that z real and Re $z > 1$ gives Re $F(z) > 0$ and Re $z < -1$ gives Re $F(z) < 0$, then $F(z_1) = F(z_2)$ for real z_1 and z_2 with $|z_1| > 1$, $|z_2| > 1$ gives $z_1 z_2 > 0$. But $F(z_1) = F(z_2)$, $z_1 \neq z_2$, yields

$$0 = (z_2 - z_1) \int_{-1}^{+1} \frac{d\phi(t)}{(z_1 - t)(z_2 - t)}$$

which is a contradiction since the integral is real and positive.

The above theorem and the consideration that a univalent function of a univalent function is univalent yield the following corollary.

COROLLARY 2.3. *If $\phi(t)$ is nondecreasing and bounded for $a \leqq t \leqq b$, then $G(z)$ given by*

$$G(z) = \int_a^b \frac{d\phi(t)}{z - t}, \qquad b > a, \tag{2.4}$$

defined and analytic for z exterior to the cut $a \leqq \xi \leqq b$ is univalent for z exterior to the circle $|w-(a+b)/2| \leqq (b-a)/2$.

For by a simple transformation and substitution we can transform the representation for $G(z)$ into the form (2.3).

3. Theorems on continued fractions which are Stieltjes transforms. The previously mentioned connection between Stieltjes transforms and continued fractions enables us now to deduce the following theorems.

THEOREM 3.1. *If $g_0, g_1, \cdots$ are real numbers satisfying $0 \leqq g_p \leqq 1$, $p = 0, 1, 2, \cdots$, then the function $G(x)$ given by*

$$G(x) = \frac{1}{1 +} \frac{(1 - g_0)g_1 x}{1 \quad +} \frac{(1 - g_1)g_2 x}{1 \quad +} \cdots \tag{3.1}$$

is univalent for Re $x \geqq -1$, $x \neq -1$. *Moreover there exist functions of the form* (3.1) *not univalent in any larger region including the one given above.*

Wall has shown [9] that the continued fraction giving $G(x)$ converges uniformly over every finite closed region whose distance from the real interval $-\infty \leqq k \leqq -1$ is positive and that (3.1) has the representation (2.1) [10], hence the result follows from Theorem 2.1.

Consider $G(x)$ given by

$$G(x) = \frac{1}{1 +} \frac{ax}{1 +} \frac{(1/2 - \epsilon)x}{1 \quad +} \frac{x/2}{1}, \qquad 0 < \epsilon < 1/2, 0 < a < 2\epsilon.$$

If $G(x_1) = G(x_2)$, then $(x_1 - x_2)(2 + (1 - \epsilon)x_1x_2 + x_1 + x_2) = 0$. Now for x_2 in the half-plane Re $x \geqq -1$ and x_1 exterior to it, the second factor above is zero. This yields x_1 as a linear fractional expression in x_2. This expression when considered as a transformation maps the line

Re $x_2 = -1$ into the circle in the x_1-plane

$$\left| x_1 + 1/2\epsilon(1-\epsilon) \right| = (1-2\epsilon)/2\epsilon(1-\epsilon).$$

Thus for every point x_1 with Re $x_1 < -1$, there exists a value of ϵ such that $G(x_1) = G(x_2)$ for some x_2 with Re $x_2 = -1$, $x_2 \neq -1$.

Closely related to the above theorem we have the following:

THEOREM 3.2. *If $g_1, g_2, \cdots$ are real numbers satisfying $0 \leqq g_p \leqq 1$, $p = 1, 2, \cdots$, then the analytic function given by*

$$F(x) = \frac{x}{1+}\,\frac{(1-g_1)g_2x}{1\quad+}\,\frac{(1-g_2)g_3x}{1\quad+}\cdots \tag{3.2}$$

is univalent for Re $x \geqq -1$, $x \neq -1$. *Moreover there exist functions of the form* (3.2) *not univalent in any larger region including the above region.*

From the results of Wall cited above the continued fraction converges for Re $x \geqq -1$, $x \neq -1$, and has the representation (2.2). This theorem is also a consequence of Theorem 3.1 since

$$G(x) = 1/(1 + (1 - g_0)g_1F(x))$$

where $F(x)$ is of the form (3.2). Univalence of $G(x)$ implies univalence of $F(x)$ and conversely.

The example following Theorem 3.1 shows the "bestness" of the region of univalence for this class of functions.

Theorems 3.1 and 3.2 show that the class of functions $G(x)$ of (3.1) is univalent in Re $x \geqq -1$, $x \neq -1$, and the class $xG(x)$ is univalent in this same region.

It is interesting to note that if $G(x)$ is of the form (3.1) and if we write $G_0^*(x) = g_0G(x)$, and if $\operatorname{lub}_{|z|<1} |G_0^*(x)| \leqq 1$, then we may apply the Schur algorithm [5] to $G_0^*(x)$ to define recursively a set of functions $G_p^*(x)$, $p = 1, 2, \cdots$. Then Wall has shown [9] that each $G_p^*(x)$ has a continued fraction expansion of the same form as $G_0^*(x)$. Hence the above theorem gives that each of these Schur functions is univalent in Re $x \geqq -1$, $x \neq -1$.

The function

$$F(z) = \frac{1}{z\;-}\,\frac{a_1^2}{z}$$

whose derivative vanishes at $z = \pm a_1 i$, where a_1 may be unity, yields an upper bound for circular neighborhoods of univalence about infinity for real J-fractions whose parameters form a chain sequence.

We show in the following theorem that this upper bound is also the lower bound.

THEOREM 3.3. *If the sequence of real numbers* $\{a_p^2\}$ *is a chain sequence, then the function* $F(z)$ *given by*

$$F(z) = \frac{1}{z -}\ \frac{a_1^2}{z -}\ \frac{a_2^2}{z -}\ \cdots \tag{3.3}$$

is univalent for $|z| > 1$. *Moreover there exist functions of the form* (3.3) *not univalent in any larger circular neighborhood of infinity.*

The conditions of the Theorem assure us that $F(z)$ is defined and analytic for $|z| > 1$ [10]. Also $F(z)$ has the representation (2.3), so that Theorem 2.3 gives the desired result.

4. **Theorems on continued fractions.** The function $F(x)$ given by

$$F(x) = \frac{x}{1 +}\ \frac{a_1 x}{1 +}\ \frac{a_2 x}{1 +}\ \cdots,$$

where $a_p = 1/4$, $p = 1, 2$, shows that the region of univalence for this class of functions has upper bound $|x| < 1$. We give an estimate of the lower bound.

THEOREM 4.1. *If* $a_1, a_2, \cdots$ *are complex numbers satisfying* $|a_p| \leqq 1/4$, $p = 1, 2, \cdots$, *then the function* $F(x)$ *given for* $|x| < 1$ *by*

$$F(x) = \frac{1}{1 +}\ \frac{a_1 x}{1 +}\ \frac{a_2 x}{1 +}\ \cdots \tag{4.1}$$

is univalent for $|x| < 4(2)^{1/2}/(3+2(2)^{1/2})$. (*The author conjectures that* $F(x)$ *is univalent for* $|x| < 1$.)

Now $F(x)$ is analytic for $|x| < 1$. From results on the value region problem for this class of functions [5] it is easy to show that for $x \lessgtr \rho < 1$, $|F(x)| \leqq 2(1-(1-\rho)^{1/2})/\rho$. If we write $F(x) \equiv F_0(x)$ and $F_k(x) = 1/(1 + a_{k+1} x F_{k+1}(x))$, $k = 0, 1, \cdots$, we see that each $F_k(x)$, $k \geqq 1$, has a continued fraction representation of the same form as $F_0(x)$ and hence the above estimate is applicable to each $F_k(x)$.

If $F_0(x_1) = F_0(x_2)$, $x_1 \neq x_2$, $|x_1| < 1$, $|x_2| < 1$, then $x_1 F_1(x_1) = x_2 F_1(x_2)$ and from a result of Montel [4] we must have $|x_1| = |x_2|$ $(= \rho < 1)$. When $F_1(x)$ is written as a transformation in $F_2(x)$, the above equality becomes

$$x_1 - x_2 = a_2 x_1 x_2 [F_2(x_1) - F_2(x_2)]. \tag{4.2}$$

Now the difference $F_k(x_1) - F_k(x_2)$ may be written

$$F_k(x_1) - F_k(x_2) = a_{k+1}F_k(x_1)F_k(x_2)F_{k+1}(x_1)F_{k+1}(x_2) \cdot \left[(x_2 - x_1) + a_{k+2}x_1x_2(F_{k+2}(x_1) - F_{k+2}(x_2))\right]. \tag{4.3}$$

When we use (4.3) in (4.2) $n+1$ times successively and apply absolute values and the triangle inequality we obtain

$$\begin{aligned} |x_1 - x_2| \leq |x_1 - x_2| &\sum_{k=1}^{n} |x_1x_2|^n \prod_{p=2}^{2k+1} |a_pF_p(x_1)F_p(x_2)| \\ &+ |x_1x_2|^n \left(\prod_{p=2}^{2n+1} |a_pF_p(x_1)F_p(x_2)|\right) \\ &\cdot |F_{2n+2}(x_1) - F_{2n+2}(x_2)|. \end{aligned} \tag{4.4}$$

If we apply the estimate of $|F_p(x)|$ to the nonhomogeneous member of (4.4), we find that it becomes zero as n becomes infinite, provided $|x_1| = |x_2| = \rho \leq 4(2)^{1/2}/(3+2(2)^{1/2})$.

Thus we may allow n to become infinite in (4.4) and the inequality prevails; we obtain, if the continued fraction does not terminate,

$$\begin{aligned} |x_1 - x_2| &\leq |x_1 - x_2| \sum_{k=1}^{\infty} |x_1x_2|^k \prod_{p=2}^{2k+1} |a_pF_p(x_1)F_p(x_2)| \\ &\leq |x_1 - x_2| \sum_{k=1}^{\infty} \rho^{2k}\left(\frac{1-(1-\rho)^{1/2}}{\rho}\right)^{4k} \\ &= |x_1 - x_2| \frac{(1-(1-\rho)^{1/2})^4}{\rho^2 - (1-(1-\rho)^{1/2})^4}. \end{aligned} \tag{4.5}$$

But if $\rho < 4(2)^{1/2}/(3+2(2)^{1/2})$, then $(1-(1-\rho)^{1/2})^4/[\rho^2-(1-(1-\rho)^{1/2})]^4 < 1$. Hence for this range of ρ, (4.5) yields $|x_1-x_2| \leq k|x_1-x_2|$ where $0<k<1$ which gives $x_1=x_2$.

If the continued fraction terminates, then only a finite number of these terms appear in the sum in the right member of (4.5) and the same conclusion applies.

If we note that (4.1) is of the form

$$F(x) = 1/1 + a_1G(x), \qquad |G(x)| \leq 2,$$

so that univalence of $F(x)$ implies univalence of $G(x)$ and conversely, we have the following corollary.

COROLLARY 4.1. *If $a_2, a_3, \cdots$ are complex numbers satisfying $|a_p| \leq 1/4$, $p=2, 3, \cdots$, then the function $G(x)$ given by*

$$G(x) = \frac{x}{1} + \frac{a_2x}{1} + \frac{a_3x}{1} + \cdots \tag{4.6}$$

is univalent for $|x| < 4(2)^{1/2}/(3+2(2)^{1/2})$.

This theorem and its corollary show that the class of functions $F(x)$ of (4.1) is univalent in $|x| < 4(2)^{1/2}/(3+2(2)^{1/2})$ while the class $xF(x)$ is univalent in the same region. If the numbers a_p are restricted to be real and positive, then $\{a_p\}$ is a chain sequence so that Theorem 3.1 is applicable and $F(x)$ and $xF(x)$ are univalent in $|x| < 1$.

We now apply the iterative method of the preceding theorem to a particular type of J-fraction.

THEOREM 4.2. *If a_p and b_p are complex numbers with the numbers a_p satisfying $|a_p| < M/3$, and if the function $F(z)$ given by*

$$F(z) = \frac{1}{z+b_1} - \frac{a_1^2}{z+b_2} - \frac{a_2^2}{z+b_3} - \cdots \tag{4.7}$$

is positive definite for Im $z>0$, *then $F(z)$ is univalent for* Im $z > M(2)^{1/2}/3$.

From a result of Dennis and Wall [1], $F(z)$ is analytic for Im $z>0$. If we write

$$F(z) \equiv F_0(z) = \frac{1}{z+b_1 - a_1^2 F_1(z)},$$

$$F_p(z) = \frac{1}{z+b_{p+1} - a_{p+1}^2 F_{p+1}(z)}, \qquad p = 0, 1, \cdots,$$

then $F_0(z_1) = F_0(z_2)$ yields

$$|z_1 - z_2| = |a_1^2|\,|F_1(z_1) - F_1(z_2)|. \tag{4.8}$$

An expansion of $F_p(z_1) - F_p(z_2)$ followed by the application of absolute values and the triangle inequality gives

$$\begin{aligned} |F_p(z_1) - F_p(z_2)| \leqq\ & |F_p(z_1)F_p(z_2)|\,|z_1 - z_2| \\ & + |a_{p+1}^2 F_p(z_1)F_p(z_2)|\,|F_{p+1}(z_1) - F_{p+1}(z_2)|. \end{aligned}$$

If this process of expansion is iterated k times in (4.8) we obtain

$$\begin{aligned} |z_1 - z_2| \leqq\ & |z_1 - z_2| \sum_{n=1}^{k}\left(\prod_{p=1}^{n} |a_p^2 F_p(z_1)F_p(z_2)|\right) \\ & + |a_{k+1}^2|\,|F_{k+1}(z_1) - F_{k+1}(z_2)|\left(\prod_{p=1}^{k} |a_p^2 F_p(z_1)F_p(z_2)|\right). \end{aligned} \tag{4.9}$$

Now since $F_0(z)$ is positive definite for Im $z>0$, the numbers a_p and b_p are such that $\alpha_p = \text{Im } a_p$ and $\beta_p = \text{Im } b_p$ satisfy the necessary and sufficient conditions for positive definiteness [11]:

(a) $\beta_p \geqq 0, \qquad p = 1, 2, \cdots,$

(4.10) (b) there exists numbers $g_0, g_1, \cdots$ such that

$$0 \leqq g_p \leqq 1 \quad \text{and} \quad \alpha_p^2 = \beta_p\beta_{p+1}(1 - g_{p-1})g_p.$$

Then each $F_p(z)$ is such that its continued fraction representation has partial numerators and denominators satisfying (4.10) and is thus positive definite for Im $z>0$. From results on positive definite continued fractions we have $|F_p(z)| <3/M(2)^{1/2}$ and $|F_p(z_1)-F_p(z_2)| <3/M(2)^{1/2}$ for $\eta=\text{Im } z>M(2)^{1/2}/3$.

When these estimates are applied to the nonhomogeneous member of (4.9), we see that it has limit zero as k becomes infinite provided z_1 and z_2 have imaginary parts greater than $M(2)^{1/2}/3$. Thus we may allow k to become infinite in (4.9) and the inequality holds in the limit. We obtain, if the continued fraction does not terminate

$$\begin{aligned} |z_1 - z_2| &\leqq |z_1 - z_2| \sum_{n=1}^{\infty}\left(\prod_{p=1}^{n} | a_p^2F_p(z_1)F_p(z_2) |\right) \\ &< |z_1 - z_2| \sum_{n=1}^{\infty} \frac{1}{2^n} = |z_1 - z_2|. \end{aligned} \tag{4.11}$$

Hence there is a contradiction unless $z_1=z_2$. If the continued fraction terminates, the inequalities in (4.11) are strengthened and the same conclusion applies.

Now if (4.7) is a bounded J-fraction, that is, the numbers b_p also satisfy $|b_p| <M/3$, $p=1, 2, \cdots$, then there exists a convex region K in the z-plane exterior to which (4.7) converges [10]. This region is obtained by performing a rotation and suitable translation of (4.7), depending upon the rotation, and noting that (4.7) is positive definite after such a transformation. This rotation and translation leave unchanged the absolute values $|a_p^2|$ in (4.7).

Hence after each rotation and translation we have, if ζ is the new variable, that the continued fraction is univalent for Im $\zeta > M(2)^{1/2}/3$. We obtain for all possible rotations a set of half-planes whose inner envelope is a region similar to the region K but at a distance $M(2)^{1/2}/3$ from K. Hence the following theorem:

THEOREM 4.3. *If $F(z)$ of* (4.7) *is given by a bounded J-fraction with $|a_p| <M/3$, $|b_p| <M/3$, $p=1, 2, \cdots$, and if K is the convex region for this J-fraction with the boundary of K given by* Im $e^{i\theta}z\leqq Y(\theta)$, *$Y(\theta)$ real, then $F(z)$ is locally univalent exterior to the region R*: Im $e^{i\theta}z\leqq Y(\theta)+M(2)^{1/2}/3$.

We may apply the same argument as in Theorem 4.2 using the same bounds for $|F_p(z_1)|$, $|F_p(z_2)|$ provided z_1 and z_2 are in R.

Corollary 4.1 is related to a type of bounded J-fraction in which the bound on the numbers a_p is different from the bound on the numbers b_p. For the even part of the S-fraction

$$\frac{x}{1+}\,\frac{a_2x}{1+}\,\frac{a_3x}{1+}\cdots, \qquad |a_n| \leqq 1/4, n = 2, 3, \cdots,$$

may be written as a J-fraction of the form

$$\frac{1}{z+a_2-}\,\frac{a_2a_3}{z+(a_3+a_4)-}\cdots\frac{a_{2n}a_{2n+1}}{-z+(a_{2n+1}+a_{2n+2})-}\cdots$$

where $z=1/x$. Consequently such a J-fraction is univalent for $|z|>1/(4(2)^{1/2})/(3+2(2)^{1/2})=(3(2)^{1/2}/4+1)/2$. But this J-fraction is of the form (4.7) with $|a_p|\leqq 3/2$, $|b_p|\leqq 3/4$. We now demonstrate a theorem which yields precisely the same result for this J-fraction.

It should be noted however that there exist J-fractions of the class covered by this theorem which are not the even part of any S-fraction.

THEOREM 4.4. *If a_p and b_p are complex numbers satisfying $|a_p|<N/3$, $|b_p|<M/3$, $p=1, 2, \cdots$, then the function $F(z)$ given by*

$$F(z) = \frac{1}{z+b_1-}\,\frac{a_1^2}{z+b_2-}\,\frac{a_3^2}{z+b_3-}\cdots \tag{4.12}$$

is univalent for $|z|>N(2)^{1/2}/2+M/3$.

By an equivalence transformation (4.12) may be thrown into the form

$$F(z) = \frac{1}{z+b_1}\left(\frac{1}{1+}\,\frac{-a_1^2/(z+b_1)(z+b_2)}{1}\right.$$
$$\left.+\frac{-a_2^2/(z+b_2)(z+b_3)}{1+}\cdots\right) = \frac{1}{z+b_1}G(z).$$

For $|z|\geqq(2N+M)/3$ the partial numerators of the continued fraction $G(z)$ have modulus less than $1/4$, hence $G(z)$ converges to an analytic function and satisfies $|G(z)|\leqq 2$. Also from value region results for this type of function [5] we have that for $|z|>(2N+M)/3$

$$|G(z)| \leqq (3|z|-M)(3|z|-M-((3|z|-M)^2-4N^2)^{1/2})/2N^2,$$

and

$$|F(z)| \leqq 3(3|z| - M - ((3|z| - M)^2 - 4N^2)^{1/2})/2N^2.$$

We write $F(z) \equiv F_0(z)$ and $F_p(z) = 1/(z + b_{p+1} - a_{p+1}^2 F_{p+1}(z))$, $p = 0, 1, 2, \cdots$, then if $F(z_1) = F(z_2)$ we have

$$|z_1 - z_2| = |a_1^2| \, |F_1(z_1) - F_1(z_2)|.$$

The difference $F_p(z_1) - F_p(z_2)$ may be written

$$\begin{aligned} |F_p(z_1) - F_p(z_2)| \\ = |F_p(z_1)F_p(z_2)| \, |z_2 - z_1 + a_{p+1}^2(F_{p+1}(z_1) - F_{p+1}(z_2))|. \end{aligned}$$

Hence after n applications of this expansion followed by the triangle inequality we obtain

$$\begin{aligned} &|z_1 - z_2| \\ &\leqq |z_1 - z_2| \sum_{k=1}^{n} \left(\prod_{p=1}^{k} |a_p^2 F_p(z_1) F_p(z_2)| \right) \\ &\quad + \prod_{p=1}^{n} |a_p^2 F_p(z_1) F_p(z_2)| \, |a_{n+1}^2| \, |F_{n+1}(z_1) - F_{n+1}(z_2)|. \end{aligned} \tag{4.13}$$

Since each $F_p(z)$ is of the same form as $F(z)$, each $F_p(z)$ satisfies the same estimate so that

$$|a_p^2 F_p(z_1) F_p(z_2)| \leqq (3|z| - M - ((3|z| - M)^2 - 4N^2)^{1/2})/4N^2$$

for $|z| > (2N+M)/3$. In particular for $|z| > N(2)^{1/2}/2 + M/3$ we have $|a_p^2 F_p(z_1) F_p(z_2)| < 1/2$, $p = 0, 1, \cdots$. Thus for this range of z the second member of the right of (4.13) has limit zero as n becomes infinite so that as we pass to the limit in (4.13) the inequality holds. We have then upon applying the estimates again

$$\begin{aligned} |z_1 - z_2| &\leqq |z_1 - z_2| \sum_{k=1}^{\infty} \left(\prod_{p=1}^{k} |a_p^2 F_p(z_1) F_p(z_2)| \right) \\ &< |z_1 - z_2| \sum_{k=1}^{\infty} \frac{1}{2^k} = |z_1 - z_2|, \end{aligned}$$

for z_1 and z_2 satisfying $|z| > N(2)^{1/2}/2 + M/3$. Thus $z_1 = z_2$. As previously, the above inequalities hold if the continued fraction terminates.

Now the convex region for the continued fraction of Theorem 4.3 lies entirely within the circle $|z| \lessapprox (2N+M)/3$. Hence Theorem 4.3 and Theorem 4.4 are independent in that each may yield regions of univalence not included in the region given by the other.

5. Theorems on functions giving maps which are stars.

THEOREM 5.1. *The function* $F(x)$ *of* (2.2) *maps the region* $|x| \leqq r < (2)^{1/2}/2$ *onto a region which is a star with respect to the origin.*

Now $F(x)$ is univalent for $|x| < 1$ hence a sufficient condition for a map of $|x| < r$ which is a star is $\operatorname{Re} xf'(x)/f(x) \geqq 0$, $|x| = r$.

But

$$\operatorname{Re} x\frac{f'(x)}{f(x)} = \left|\int_0^1 \frac{d\phi(t)}{1+xt}\right|^{-2}\left\{\int_0^1 \frac{(1+ut)^2-(vt)^2}{|1+xt|^4}\,d\phi(t) \cdot \int_0^1 \frac{1+ut}{|1+xt|^2}\,d\phi(t) + 2\int_0^1 \frac{v^2t(1+ut)}{|1+xt|^4}\,d\phi(t)\int_0^1 \frac{t}{|1+xt|^2}\,d\phi(t)\right\}. \tag{5.1}$$

For $|x| < 1$ and $u \geqq 0$ each of the integrals in (5.1) is non-negative while for $|x| = 2^{1/2}/2$ we have $1+ut \geqq |vt|$ for $0 \leqq t < 1$ so that again all integrals in (5.1) are positive.

THEOREM 5.2. *The function* $G(x)$ *of* (4.6) *maps the region* $|x| \leqq \rho \leqq 8/9$ *onto a region star-like with respect to the origin.*

Now $G(x)$ is univalent for $|x| < 8/9$, so that the condition used above is applicable. Write $G(x) \equiv G_1(x) = x/(1+a_2G_2(x))$ and generally $G_k(x) = x/(1+a_{k+1}G_{k+1}(x))$, $k = 1, 2, \cdots$.

For $|x| \leqq \rho < 1$ each $G_k(x)$ satisfies $|G_k(x)| \leqq 2(1-(1-\rho)^{1/2})$. Thus

$$x\frac{G'_{k-1}(x)}{G_{k-1}(x)} = 1 - a_kG_{k-1}(x)G'_k(x). \tag{5.2}$$

Now suppose that $G_1(x)$ is given by a continued fraction terminating with nth partial numerator and is denoted by $G_1^{(n)}(x)$. Since $G_n(x) = a_nx$ we have $|a_nG_{n-1}(x)G'_n(x)| = |a_n^2G_{n-1}(x)| < 1$. If $|a_pG_{p-1}(x) \cdot G'_p(x)| \leqq 1$ for $p = n, n-1, \cdots, k+1$, then

$$\begin{aligned}
|a_kG_{k-1}(x)G'_k(x)| &= \left|\frac{a_kG_{k-1}(x)}{1+a_{k+1}G_{k+1}(x)}\right| |1 - a_{k+1}G_k(x)G'_{k+1}(x)| \\
&\leqq \left|\frac{a_kG_{k-1}(x)}{1+a_{k+1}G_{k+1}(x)}\right| [1 + |a_{k+1}G_k(x)G'_{k+1}(x)|] \\
&\leqq \frac{2(1-(1-\rho)^{1/2})^2}{\rho} \leqq 1
\end{aligned}$$

if $\rho \leqq 8/9$. Thus by induction $|a_k G_{k-1}(x) G_k'(x)| \leqq 1$, $k=2, 3, \cdots, n$. Hence from (5.2)

$$\operatorname{Re} x \frac{G_1'(x)}{G_1(x)} = 1 - \operatorname{Re} a_2 G_1(x) G_2'(x) \geqq 1 - |a_2 G_1(x) G_2'(x)| \geqq 0$$

in case $G(x)$ is a terminating continued fraction $G_1^{(n)}(x)$.

If $G(x)$ is a nonterminating continued fraction then it is the limit of a uniformly convergent sequence of terminating continued fractions $G(x) = \lim_{n\to\infty} G_1^{(n)}(x)$. Hence

$$\operatorname{Re} x \frac{G'(x)}{G(x)} = \lim_{n\to\infty} \operatorname{Re} x \frac{G_1^{(n)\prime}(x)}{G_1^{(n)}(x)} \geqq 0.$$

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