THE DERIVATIVE OF A MEROMORPHIC FUNCTION

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1. It has been shown by Valiron [2] and Whittaker [3] that the derivative of a meromorphic function of finite order is of the same order as the function itself. This result, as pointed out by Whittaker, is equivalent to the following.

**Theorem.** If \( f(z) \) and \( g(z) \) are two integral functions of orders \( \rho_1 \) and \( \rho_2 \) with \( \rho_1 > \rho_2 \), then \( f'(z)g(z) - f(z)g'(z) \) is of order \( \rho_1 \).

The proofs given by Valiron and Whittaker depend on meromorphic function theory, but in this paper I shall give a proof of the above theorem which depends entirely on integral function theory.

2. A number of lemmas are required and no proof will be given for the first of these as it is already well known.

**Lemma 1** [1, p. 102]. Except for an exceptional set of intervals within which the variation of \( \log r \) is finite

\[
zf'(z) = Nf(z)\{1 + o(1)\}, \quad |z| = r,
\]

where \( |f(z)| = M(r, f) \) and \( N = N(r, f) \).

**Lemma 2.** There is an infinite sequence \( \{\lambda_i\}_{1}^{\infty} \) such that if \( \lambda_i \leq r \leq \lambda_i^\alpha \), \( \alpha = (\rho_1 - \epsilon/2)(\rho_1 - \epsilon)^{-1} \), then

\[
\log N(r, f) \geq (\rho_1 - \epsilon) \log r.
\]

From the result [1, p. 33]

\[
\lim \sup_{r \to \infty} \frac{\log N(r, f)}{\log r} = \rho_1
\]

it follows that there is an infinite sequence \( \{\lambda_i\} \) such that

\[
\{\log N(\lambda_i, f)/\log \lambda_i \} / \log \lambda_i \geq \rho_1 - \epsilon/2.
\]

Since \( N(r, f) \) is a nondecreasing function of \( r \) this means that

\[
\{\log N(r, f)/\log r \} / \log r \geq \rho_1 - \epsilon
\]

provided \( (\rho_1 - \epsilon/2) \log \lambda_i \geq (\rho_1 - \epsilon) \log r \), which gives the lemma.

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Lemma 3. Except when \( r \) lies in a set of intervals of total finite length

\[ |f'(z)/f(z)| < r^{\rho_1+\varepsilon}, \quad |z| = r, \quad \varepsilon > 0. \]

If \( p \) is the genus of \( f(z) \) then

\[ f'(z)/f(z) = \rho(z) + \sum_{1}^{\infty} 2^{p} \{(z - a_{n})a_{n}^{p}\}^{-1} \]

where \( \{a_{n}\} \) is the sequence of zeros of \( f(z) \) and \( \rho(z) \) is a polynomial whose degree does not exceed \( \rho_{1}-1 \). Hence

\[ |f'(z)/f(z)| \leq O(r^{\rho_{1} - 1}) + r^{p} \sum_{1}^{\infty} \{|r - r_{n}| r_{n}^{p}\}^{-1}. \]

Given \( \sigma \) and \( k > 1 \) we suppose \( \sigma k^{-1/2} \leq r \leq \sigma k^{1/2} \) and consider the above sum in three parts \( \sum_{1}, \sum_{2}, \sum_{3} \) such that \( r_{n} < \sigma k^{-1}, \sigma k^{1} \leq r_{n} \leq \sigma k, \)

\( r_{n} > \sigma k \) respectively. Using the fact that \( \rho + 1 \) exceeds the exponent of convergence of the zeros, the sums \( \sum_{1} \) and \( \sum_{3} \) can be shown to satisfy

\[ \sum_{1} = O(r^{p}), \quad \sum_{3} = O(r^{p}). \]

Integrating \( \sum_{2} \) with respect to \( r \) over \((\sigma k^{-1/2}, \sigma k^{1/2})\), excluding intervals of length \( 2\eta \) centered on each \( r_{n} \), where \( \eta = (\sigma k^{-1/2})^{-h}, h > \rho_{1} \), we get

\[ \int \sum_{2} dr = O(\sigma^{h+\varepsilon} \log \sigma) \]

provided, given \( \delta > 0 \), we choose \( \sigma \) sufficiently large. The range of integration is of length not less than \((k^{1/2} - k^{-1/2})\sigma - 2n(k\sigma)(\sigma k^{-1/2})^{-h}, \)

which is \((k^{1/2} - k^{-1/2})\sigma - O(1)\). This means that \( \sum_{2} < r^{\rho_{1}+\varepsilon} \) except in a set of intervals whose lengths in sum do not exceed \( O(\sigma^{-d} \log \sigma) \).

Combining these results gives the lemma.

3. The proof of the theorem will now be given. From the lemmas it follows that we can find an infinite sequence of \( r \) such that, with \(|z| = r, \)

\[ \log N(\sigma, f) \geq (\rho_{1} - \varepsilon) \log \sigma, \quad \rho^{a} \leq \sigma \leq r, \quad \alpha = (\rho_{1} - \varepsilon)(\rho_{1} - \varepsilon/2)^{-1}, \]

\[ zf'(z)/f(z) = N(r, f) \{1 + o(1)\}, \quad |f(z)| = M(r, f), \]

\[ |g'(z)/g(z)| < r^{\rho_{1}+\varepsilon}. \]

Also [1, p. 32]
\[
\log M(r, f) \sim \log \mu(r, f) \\
= K + \int_1^r x^{-1} N(x, f) \, dx \\
> \int_\rho r x^{-1} N(x, f) \, dx \\
> \rho^{\rho - \epsilon}
\]

if \( r \) is large enough. Finally
\[
|f'(z)g(z) - f(z)g'(z)| > |f(z)g(z)| \left\{ \left| \frac{f'(z)}{f(z)} \right| - \left| \frac{g'(z)}{g(z)} \right| \right\} \\
> \exp (\rho^{\rho - \epsilon} - \rho^{\rho + \epsilon}) (\rho^{\rho - \epsilon - \epsilon} - \rho^{\rho + \epsilon})
\]

where use has been made of a result of Borel [1, p. 57]. Thus the theorem follows if \( \rho_1 - 1 > \rho_2 \). When this is not the case we choose an integer \( n \) so that \( n\rho_1 - 1 > n\rho_2 \) and define \( F(z) = f(z^n) \), \( G(z) = g(z^n) \). Then
\[
F'(z)G(z) - F(z)G'(z) = nz^{n-1} \left\{ f(z^n)g(z^n) - f(z^n)g'(z^n) \right\}
\]
and by the previous result \( F'(z)G(z) - F(z)G'(z) \) is of order \( n\rho_1 \). Hence \( f'(z)g(z) - f(z)g'(z) \) is of order \( \rho_1 \) which proves the theorem in this case also.

4. As pointed out by Whittaker the above result is the best possible, as can be seen by taking \( f(z) = \cos z \) and \( g(z) = \sin z \). However, it would be of interest to know if the theorem was true for functions of the same order if the type of \( g(z) \) were less than that of \( f(z) \).

It would also be of interest to know what can be said about the order of \( f'(z)g(z) + f(z)h(z) \) when \( g(z) \) and \( h(z) \) have orders less than that of \( f(z) \).

References

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