SEMIGROUPS IN COMPACT GROUPS

FRED B. WRIGHT

In the theory of one-parameter semigroups, the parameter family is usually an open semigroup of the real or complex numbers (under addition). Thus it is fundamental to investigate the structure of such semigroups. For a large class of these the structure is well known. These are the angular semigroups [3, Definition 7.6.1]; that is, those open semigroups which have the identity element 0 as a limit point. A complete discussion of angular semigroups of the real line and Euclidean 2-space will be found in [3, Chapter VII] and [4]. Furthermore, these methods of Hille and Zorn can be extended to a characterization of angular semigroups in Euclidean $n$-space $E^n$ [12].

In this paper the structure of all open semigroups in any compact topological group is determined. We can then extend this result to a classification of the angular semigroups of any abelian topological group $H$ which contains a compact open subgroup $K$. Coupling this with the results for Euclidean $n$-space, we obtain an essentially complete description of the angular semigroups of an arbitrary locally compact group.

For compact groups, the result is remarkably simple.

**Theorem I.** Let $K$ be any compact group, and let $S$ be a semigroup in $K$. Then $S$ is necessarily a closed subgroup of $K$ under either of the following conditions: (1) $S$ is closed, (2) $S$ is open.

**Proof.** The case where $S$ is closed is already well known, and is contained in a theorem on topological semigroups due to several authors [2; 5; 6; 8; 9; 13]. The facts are these: $S$ is a topological semigroup, or mob, and is compact. Then $S$ must contain idempotents, and $S$ is a topological group if and only if there is an identity element and there are no other idempotents in $S$. It is clear that these conditions are satisfied in the present case, and hence $S$ is a group.

The case where $S$ is open reduces to this. For, let $H = \overline{S}$; then $H$ is a closed semigroup, and therefore a group. If the identity element 1 of $K$ is in $S$ and isolated in $S$, then it is isolated in $K$, and therefore $K$ is discrete and finite. Either the standard algebraic result [15, Theorem 1, p. 3] or the above remarks then imply that $S$ is a group. Otherwise, 1 is a limit point of $S$, and $S$ is therefore angular. By [3, Theorem 7.6.2], we have $S = H^0$, where $H^0$ denotes the interior of $H$. But

---

Received by the editors April 25, 1955.

1 Written under contract N7-onr-434, Task Order III, Office of Naval Research.
any subgroup of a topological group having a nonvoid interior is an open and closed subgroup, so that \( S = H^0 = H \), q.e.d.

We may now appeal to the Zorn Category Theorem [3, Theorem 7.7.1] for the following consequence, which bears a striking resemblance to well known theorems of Banach [1, Theorem 1, p. 21] and Kuratowski [3, Theorem 1.8.1].

**Theorem II.** Let \( K \) be a compact group and let \( S \) be a semigroup in \( K \) which is of the second category at the identity 1 of \( K \) and which satisfies the condition of Baire. Then \( S \) is a subgroup of \( K \) which is both open and closed.

**Proof.** Zorn's theorem states that the interior \( S^0 \) of \( S \) is dense in \( S \) and that \( S^0 \) is the interior of \( S \). But \( S^0 \) is clearly a semigroup, and is not empty since \( S \) is not. Then Theorem I applies: \( S^0 = S = \overline{S} \).

Slightly more general versions of the category theorem will be found in [7; 10].

A further result, reminiscent of the situation in locally compact groups, is the following immediate consequence of Theorem I.

**Theorem III.** In a compact group \( K \), any semigroup \( S \) which is locally compact in its induced topology is a subgroup of \( K \) which is both open and closed.

Now let \( H \) be an abelian topological group containing a compact open subgroup \( K \), and let \( S \) be any open semigroup in \( H \). If \( S \cap K \) is not empty, then \( S \cap K \) is an open subgroup of \( K \), and hence of \( H \), by Theorem I. On the other hand, if \( S \cap K \) is empty, then \( S \) is not angular, since \( K \) is a neighborhood of the identity 1 of \( H \). If \( S \neq S \cap K \), let \( x \in S, x \notin K \). Since \( S \) is a semigroup we have \( x(S \cap K) \subseteq S \). This shows that if \( S^* \) is the image of \( S \) in \( H/S \cap K \), then the complete inverse image of \( S^* \) is simply \( S \) again. Clearly \( S^* \) contains the identity of \( H/S \cap K \). Conversely, if \( K_1 \) is any open subgroup of \( K \), and if \( S^* \) is any semigroup in the (discrete) group \( H/K_1 \) which contains the identity, then the complete inverse image of \( S^* \) in \( H \) is an angular semigroup of \( H \).

If \( G \) is any locally compact abelian group, then there exists a direct product decomposition \( G = E^* \times H \) of \( G \), where \( E^* \) is a Euclidean space of unique dimension \( n \) and where \( H \) is a group containing a compact open subgroup \( K \) [11; 14]. Then an angular semigroup of \( G \) is the direct product of angular semigroups in \( E^* \) and \( H \). The above discussion, coupled with the results for Euclidean spaces, thus gives an essentially complete description of the angular semigroups of \( G \).
SEMIGROUPS IN COMPACT GROUPS

REFERENCES


TULANE UNIVERSITY OF LOUISIANA