ON A PROPERTY OF MONOTONE AND CONVEX FUNCTIONS

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We shall deal with real functions which have continuous second derivatives in some open interval $(a, b)$, $-\infty \leq a < b \leq \infty$. The interval of definition of $f(x)$ is denoted by $I(f)$. $\phi(x)$ is called convex (from below) if $\phi''(x) \geq 0$, concave if $\phi''(x) \leq 0$ in $I(\phi)$.

If $\phi(x)$ is monotone increasing and convex, and $\psi(x)$ is monotone increasing and concave such that the range of $\phi(x)$ is contained in $I(\psi)$, then

$$(1) \quad f(x) = \psi(\phi(x))$$

is also monotone increasing, but usually neither convex nor concave. The question arises, under what conditions can $f(x)$ be represented in the form (1).

**Theorem 1.** If $f(x)$ is strictly monotone increasing and has a continuous second derivative in $I(f)$ then it has a representation (1).

Theorem 1 states that there is a strictly increasing concave function $\psi(u)$ with continuous second derivative such that $\psi(\psi(u)) = f(\psi(u))$ is concave. This is equivalent to

$$(2) \quad \psi'(u) = f''(\psi(u))[\psi'(u)]^2 + f'(\psi(u))\psi''(u) \leq 0,$$

or if $\phi(x)$ is the inverse of $\psi(u)$, to

$$(3) \quad f''(x)/f'(x) \leq \phi''(x)/\phi'(x).$$

Let $f''(x)$ denote $f''(x)$ if $f''(x) \geq 0$ and 0 if $f''(x) < 0$. Consider the function

$$(4) \quad \phi_0(x) = \int_d^x e^{\phi(y)} dy$$

where $d$ is any fixed number in $(a, b)$ and

$$(5) \quad \rho(y) = \int_d^y [f''(t)/f'(t)] dt.$$

Clearly $\phi_0(x) > 0$ and

Received by the editors February 21, 1955 and, in revised form, May 26, 1955.

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\( \phi''(x)/\phi'(x) = f''(x)/f'(x) \)

so that (3) is satisfied, also

\( \phi''(x) \geq 0. \)

This proves the theorem.\(^1\)

If \( f(x) \) is bounded and \( I(f) \) is finite, the question comes up whether \( \phi(x) \) itself can be chosen to be bounded. This is answered by

**Theorem 2.** If \( f(x) \) is bounded, strictly increasing and has a continuous second derivative in \((a, b)\), then it can be represented in the form

\[ (1) \]

with bounded \( \phi(x) \) if and only if the integral

\[ \int_a^b e^p(y) dy \]

converges, where \( a < d < b \) and \( p(y) \) is the function defined under (5).

We shall see presently that boundedness of \( f(x) \) does not necessarily imply finiteness of (8).

To prove Theorem 2 we first note that, by (4), \( \phi_0(x) \) is bounded from above if (8) is finite, and also bounded from below if \( f(x) \) is bounded since

\[
\phi_0(x) \geq \int_a^x \left\{ \exp \int_a^y \left[ f''(t)/f'(t) \right] dt \right\} dy
\]

by (4) and (5).

Suppose now that (8) diverges, so that \( \phi_0(x) \) is unbounded from above, and let \( \phi_1(x) \) be any function which has the properties (3) and (7). By taking a suitable linear combination \( \phi(x) = c_0 \phi_0(x) + c_1 \) we can achieve that \( \phi(d) = 0, \phi'(d) = 1 \). Now from (3) and (6),

\[ \frac{d}{dx} \log \phi'(x) \geq \frac{d}{dx} \log \phi'_0(x) \]

which implies

\[ \phi'(x) \geq \phi'_0(x), \quad \phi(x) \geq \phi_0(x) \quad \text{for } x > d. \]

This shows that \( \phi_0(x) \) is in a sense the "least convex" among all possible solutions and that \( \phi(x) \), hence also \( \phi_1(x) \), is unbounded.

\(^1\) I am indebted to G. Lorentz for a substantial shortening of the original argument. His proof, which is reproduced above, contributed greatly to a simplified treatment of another part of the paper.
Theorems 1 and 2 have obvious dual formulations.

**Theorem 1.** Under the same conditions as in Theorem 1, \( f(x) \) can be represented in the form

\[
(1^*) \quad f(x) = \phi_1(\psi(x))
\]

where \( \phi_1 \) is convex and \( \psi \) concave.

**Theorem 2.** If \( f(x) \) is as in Theorem 2, then it can be represented in the form \( (1^*) \) with bounded \( \psi(x) \) if and only if

\[
(8^*) \quad \int_a^d e^{\psi(y)} \, dy
\]

is finite, where

\[
(5^*) \quad q(y) = \int_y^d \left[ f''(t)/f'(t) \right] dt.
\]

Here \( f''(t) \) denotes \(-f''(t)\) if \( f''(t) \leq 0 \) and \( 0 \) if \( f''(t) > 0 \).

The following example shows that boundedness of \( f(x) \) does not necessarily imply finiteness of \( (8) \) or \( (8^*) \). Take \( f(x) = 2x + x^3 \sin(1/x) \) over the interval \((0, 1)\). It is easily seen that \( f''(x) \) has a zero \( x_n \) between \( 2/(2n+1)\pi \) and \( 2/(2n-1)\pi \), and

\[
\begin{align*}
    f''(x) &> 0 \quad \text{for} \quad x_{2m} < x < x_{2m-1}, \\
    f''(x) &< 0 \quad \text{for} \quad x_{2m+1} < x < x_{2m}, \quad m = 1, 2, \ldots.
\end{align*}
\]

It can also be shown easily that

\[
x_n = 1/n\pi + 2/n^3 + O(n^{-5}), \quad f'(x_n) = 2 - (-1)^n + O(n^{-2}),
\]

so that

\[
\int_{x_{2m+1}}^{x_{2m}} \left[ f''(t)/f'(t) \right] dt = - \log 3 + O(m^{-2})
\]

and

\[
q(y) = m \log 3 + O(1) \quad \text{for} \quad x_{2m+1} < x < x_{2m-1}.
\]

This shows that \( \int_0^1 e^{\phi(x)} \, dx \) is divergent.

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