

# A STRONG LIMIT THEOREM FOR GAUSSIAN PROCESSES

GLEN BAXTER

**1. Introduction and main result.** Let  $\{W(t), 0 \leq t \leq 1\}$  be the Wiener process, that is, a Gaussian stochastic process of real-valued random variables with  $E\{W(t)\} = 0$  and  $E\{W(s)W(t)\} = \min\{s, t\}$ . It was discovered independently by Levy [4] and by Cameron and Martin [1] that with probability one

$$(1) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} \left[ W\left(\frac{k}{2^n}\right) - W\left(\frac{k-1}{2^n}\right) \right]^2 = 1.$$

We demonstrate here a similar result for a large class of Gaussian processes including the Wiener process as a particular case. Notation: in the following  $\{X(t), 0 \leq t \leq 1\}$  will denote a Gaussian stochastic process of real-valued random variables with mean function  $E\{X(t)\} = m(t)$  and covariance function  $E\{X(s)X(t)\} - m(s)m(t) = r(s, t)$ .

Now assume that  $m(t)$  has a bounded first derivative for  $0 \leq t \leq 1$ . Furthermore, assume that  $r(s, t)$  is continuous in  $0 \leq s, t \leq 1$  and has uniformly bounded second derivatives for  $s \neq t$ . Let

$$D^+(t) = \lim_{s \rightarrow t^+} \frac{r(t, t) - r(s, t)}{t - s},$$

$$D^-(t) = \lim_{s \rightarrow t^-} \frac{r(t, t) - r(s, t)}{t - s},$$

$$f(t) = D^-(t) - D^+(t).$$

The uniform boundedness of the second order derivatives of  $r(s, t)$  implies the existence, boundedness, and continuity of the functions  $D^+(t)$ ,  $D^-(t)$ , and  $f(t)$  over  $0 < t < 1$ . In particular  $f(t)$  is Riemann integrable over the interval  $0 < t < 1$ . The main result can now be stated.

**THEOREM 1.** *If  $\{X(t), 0 \leq t \leq 1\}$  is a Gaussian process satisfying the assumptions of the preceding paragraph, then with probability one*

$$(2) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} \left[ X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right) \right]^2 = \int_0^1 f(t) dt.$$

Note that the existence of the first derivative of  $r(s, t)$  at  $s = t$  is not

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assumed, in fact, it is not wanted. The existence of this derivative would make  $f(t) \equiv 0$  and the right-hand side of (2) would reduce to zero, giving an uninteresting result. An important corollary to Theorem 1 holds in case  $r(s, t)$  factors into the product of a function of  $s$  and a function of  $t$ . As we shall see later there are many examples of Gaussian processes for which the covariance function has this property.

**COROLLARY.** *If  $\{X(t), 0 \leq t \leq 1\}$  satisfies the assumptions of Theorem 1, and if*

$$(3) \quad r(s, t) = \begin{cases} u(s)v(t), & s \leq t, \\ u(t)v(s), & s \geq t, \end{cases}$$

*then with probability one*

$$(4) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} \left[ X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right) \right]^2 = \int_0^1 \{u'(t)v(t) - u(t)v'(t)\} dt.$$

The proof of Theorem 1 is contained in the next section. In §3 we give some examples of the theorem and show that (1) is implied by (2) (and by (4)).

**2. Proof of Theorem 1.** Let us first assume that  $m(t) \equiv 0$ . For any positive integer  $n$  and integers  $j, k$  with  $1 \leq j, k \leq 2^n$  set

$$\begin{aligned} \Delta X_k &= X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right), \\ a_{jk} &= E\{\Delta X_j \Delta X_k\}, \\ B_n &= \sum_{k=1}^{2^n} \Delta X_k^2, \\ b_n &= 2 \sum_{j, k=1}^{2^n} a_{jk}^2. \end{aligned}$$

Observe that

$$(5) \quad E\{B_n\} = \sum_{k=1}^{2^n} a_{kk}$$

while

$$E\{B_n^2\} = \sum_{k=1}^{2^n} 3a_{kk}^2 + 2 \sum_{j>k=1}^{2^n} (a_{kk}a_{jj} + 2a_{jk}^2).$$

Thus the variance of  $B_n$  is exactly  $b_n$ . Applying Tchebychev's inequality

$$P\{ |B_n - E\{B_n\}| > n/(2^n)^{1/2} \} < 2^n b_n/n^2.$$

If we can show that  $2^n b_n$  remains bounded as  $n$  becomes infinite, an application of the Borel-Cantelli lemma will show that with probability one  $B_n - E\{B_n\}$  approaches zero as  $n$  becomes infinite.

To estimate  $2^n b_n$ , let  $M$  be a bound for the three quantities  $|\partial^2 r(s, t)/\partial s^2|$ ,  $|\partial^2 r(s, t)/\partial s \partial t|$ , and  $|\partial^2 r(s, t)/\partial t^2|$  in the range  $0 \leq s \neq t \leq 1$ . Using for  $r(s, t)$  a Taylor series expansion with remainder it can easily be shown that  $j \neq k$  implies

$$(6) \quad |a_{jk}| = |r(k/2^n, j/2^n) + r((k-1)/2^n, (j-1)/2^n) - r(k/2^n, (j-1)/2^n) - r((k-1)/2^n, j/2^n)| \leq 3M(1/2)^{2n}.$$

Also for  $k=1, 2, \dots, 2^n-1$

$$(7) \quad \begin{aligned} a_{kk} &= r(k/2^n, k/2^n) - 2r((k-1)/2^n, k/2^n) \\ &\quad + r((k-1)/2^n, (k-1)/2^n) \\ &= (1/2)^n [D^-(k/2^n) - D^+(k/2^n)] + O((1/2)^{2n}) \end{aligned}$$

where  $2^{2n}O((1/2)^{2n})$  remains bounded independent of  $k$  as  $n$  becomes infinite. The estimates in (6) and (7) give the boundedness of  $2^n b_n$ . In conjunction with (5), the estimate in (7) also implies that

$$E\{B_n\} = \sum_{k=1}^{2^n-1} \left(\frac{1}{2}\right)^n f(k/2^n) + O\left(\left(\frac{1}{2}\right)^n\right)$$

so that  $E\{B_n\}$  approaches a limit as  $n$  becomes infinite. This limit is exactly the Riemann integral of  $f(t)$  from 0 to 1. Writing

$$B_n = E\{B_n\} + [B_n - E\{B_n\}]$$

we see that with probability one  $B_n$  itself converges to the integral of  $f(t)$  over  $(0, 1)$ . This finishes the case for which  $m(t) \equiv 0$ .

If  $m(t) \neq 0$ , form a new Gaussian process  $\{\bar{X}(t), 0 \leq t \leq 1\}$  by taking  $\bar{X}(t) = X(t) - m(t)$ . From the previous arguments we know that with probability one

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} \Delta \bar{X}_k^2 = \int_0^1 f(t) dt.$$

Now by Schwarz's inequality with  $\Delta m_k = m(k/2^n) - m((k-1)/2^n)$

$$(8) \quad \left[ \sum_{k=1}^{2^n} \Delta m_k \Delta \bar{X}_k \right]^2 \leq \sum_{k=1}^{2^n} \Delta m_k^2 \cdot \sum_{k=1}^{2^n} \Delta \bar{X}_k^2.$$

The second term on the right-hand side of (8) remains bounded with

probability one. Since  $m'(t)$  exists and is uniformly bounded over  $(0, 1)$ , the sum of the  $\Delta m_k^2$  goes to zero as  $n$  becomes infinite. Thus, with probability one the left-hand side of (8) goes to zero. Finally,

$$\sum_{k=1}^{2^n} \Delta X_k^2 = \sum_{k=1}^{2^n} \Delta \bar{X}_k^2 + 2 \sum_{k=1}^{2^n} \Delta m_k \Delta \bar{X}_k + \sum_{k=1}^{2^n} \Delta m_k^2$$

so that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} \Delta X_k^2 = \lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} \Delta \bar{X}_k^2 = \int_0^1 f(t) dt.$$

This ends the proof of Theorem 1.

**3. Examples.** (a) Let us first show that (1) is included in (2) and (4). For the Wiener process  $r(s, t)$  factors into a product of a function of  $s$  and a function of  $t$  like (3); in fact,

$$r(s, t) = \begin{cases} s, & s \leq t, \\ t, & s \geq t. \end{cases}$$

In the notation of the corollary to Theorem 1,  $u(s) = s$  and  $v(t) = 1$  so (4) states that with probability one

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} \left[ X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right) \right]^2 = \int_0^1 1 dt = 1.$$

(b) For a second example<sup>1</sup> we consider a class of Gaussian processes whose covariance functions are actually Green's functions of certain simple boundary value problems.

It is necessary and sufficient in order that  $r(s, t)$  defined for  $0 \leq s, t \leq 1$  be the covariance function of a Gaussian process  $\{X(t), 0 \leq t \leq 1\}$  that

(i)  $r(s, t) = r(t, s)$ ;

(ii) if  $t_1, t_2, \dots, t_n \in [0, 1]$ , then  $(r(t_i, t_j))$  is a non-negative definite matrix.

In case  $r(s, t)$  has the form (3) where both  $u(t)$  and  $v(t)$  are non-negative for  $0 \leq t \leq 1$ , these two conditions can be reduced to just

(iii) if  $0 \leq t_1 < t_2 \leq 1$ , then  $u(t_2)v(t_1) - u(t_1)v(t_2)$  is non-negative.

To verify this last statement we need only mention that if  $r(s, t)$  has the form (3), (i) is automatically satisfied while the determinant of  $(r(t_i, t_j))$  can be explicitly evaluated:

<sup>1</sup> Some of the results here are taken from the author's doctoral dissertation which was written under the guidance of Professor Monroe Donsker at the University of Minnesota.

$$| (r(t_i, t_j)) | = u(t_1)v(t_n) \prod_{i=2}^n \{ u(t_i)v(t_{i-1}) - u(t_{i-1})v(t_i) \}.$$

If (iii) holds the determinant of  $(r(t_i, t_j))$  is non-negative for any choice of  $t_1, \dots, t_n \in [0, 1]$ . This shows (see [5, p. 103]) that  $(r(t_i, t_j))$  is non-negative definite. On the other hand, if (ii) holds for  $r(s, t)$  of the form (3), then the implications of (ii) in the case  $0 \leq t_1 < t_2 \leq 1$  give (iii).

We now show how covariance functions which factor like (3) may come from certain differential systems.

**THEOREM 2.** *Let  $p(s) > 0$ ,  $q(s) \geq 0$ , and  $p'(s)$  be continuous functions on  $0 \leq s \leq 1$ , and let  $h$  and  $H$  be non-negative extended real-valued numbers. If*

$$(9) \quad \begin{cases} \frac{d}{ds} \left\{ p(s) \frac{dy}{ds} \right\} - q(s)y = 0, \\ y(0) - hy'(0) = 0, \\ y(1) + Hy'(1) = 0 \end{cases}$$

*is incompatible, then the Green's function  $r(s, t)$  (see [2, p. 304]) is a covariance function of a Gaussian process.*

**PROOF.** It is shown in the reference above that the Green's function of system (9) has the form (3) where  $u(s)$  and  $v(s)$  are linearly independent solutions of the differential equation and satisfy, respectively, the boundary conditions at  $s=0$  and  $s=1$ . We must show that  $r(s, t)$  is non-negative and that  $u(t)v(s) - u(s)v(t) \geq 0$  for  $s \leq t$ .

To show that  $r(s, t)$  is non-negative write

$$p(s)u'(s) - p(0)u'(0) = \int_0^s u(x)q(x)dx.$$

By the non-negativity of  $p(s)$  and  $q(s)$  we deduce that  $u(s)$  cannot change sign in the interval  $0 \leq s \leq 1$  and is increasing or decreasing according as it is positive or negative. Similarly we deduce that  $v(s)$  does not change sign in the interval  $0 \leq s \leq 1$  and is increasing or decreasing according as it is negative or positive. Suppose for a while that  $u(t)v(s) - u(s)v(t)$  is positive whenever  $t > s$ . If  $u(s)v(t)$  is negative, then either  $u(s)$  is positive and  $v(t)$  is negative or vice-versa. Since either case is handled in the same way we consider only  $u(s)$  positive and  $v(t)$  negative. In that case  $t > s$  implies  $u(t)v(t) \leq u(s)v(t) < u(t)v(s)$  which contradicts the fact that  $v(t)$  is increasing in  $t$ . Thus  $u(s)v(t)$  is non-negative for  $0 \leq s, t \leq 1$ .

To prove that  $u(t)v(s) - u(s)v(t)$  is positive for  $t > s$  we write

$$(10) \quad \begin{aligned} p(t) \{ u'(t)v(s) - u(s)v'(t) \} - 1 \\ = \int_s^t q(x) \{ u(x)v(s) - v(x)u(s) \} dx. \end{aligned}$$

For  $t$  in the neighborhood of  $s$  the right-hand side of (10) is very small. Thus the derivative of  $u(t)v(s) - u(s)v(t)$  with respect to  $t$  (fixed but arbitrary  $s$ ) is positive for  $t > s$  so that the positivity of  $u(t)v(s) - u(s)v(t)$  ( $t > s$ ) is verified. This completes the proof of the theorem.

Now let  $\{X(t), 0 \leq t \leq 1\}$  be a Gaussian stochastic process whose covariance function is the Green's function of (9). By the definition of the Green's function  $u'(t)v(t) - u(t)v'(t) = 1/p(t)$  so that the corollary to Theorem 1 states that

$$(11) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} \left[ X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right) \right]^2 = \int_0^1 \frac{1}{p(t)} dt$$

with probability one. For an explicit example take the system (9) where  $p(s) = 1$ , and  $q(s) = h = H = 0$ . The system is incompatible and the Green's function is

$$r(s, t) = \begin{cases} s(1-t), & s \leq t, \\ t(1-s), & s \geq t. \end{cases}$$

The Gaussian process with this covariance function and mean function zero has been the subject of considerable study by Doob [3] and others. We have immediately for this process

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{2^n} \left[ X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right) \right]^2 = \int_0^1 \frac{1}{1} dt = 1.$$

It is easy to see that the Wiener process is another explicit example of Theorem 2 and (11).

#### REFERENCES

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY