GENERALIZED $L_2$-LAPLACIANS

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1. Given a function $F(x, y)$ in $\mathbb{R}^2$ on the plane, we shall say that $F(x, y)$ has $f(x, y)$ as a generalized $L_2$-Laplacian if

$$
8 \left[ x^{-1} \int_{D_2(0,1)} F(x + tu, y + tv) dudv - F(x, y) \right] / t^2 \rightarrow f(x, y) \text{ in the } L_2\text{-norm}
$$

where $D_2(0, 1)$ is the disc with radius 1 and center at the origin. It will be shown here that a necessary and sufficient condition that $F(x, y)$ has $f(x, y)$ as a generalized $L_2$-Laplacian is that $F(x, y)$ be locally an $L_2$-potential of $f(x, y)$, that is given any disc $D_2(0, R)$ there is a harmonic function $h_R(x, y)$ such that for almost all $(x, y)$ in $D_2(0, R)$

$$
F(x, y) = - (2\pi)^{-1} \log \left[ (x - u)^2 + (y - v)^2 \right]^{-1/2} f(u, v) dudv + h_R(x, y)
$$

where $f(x, y)$ is in $L_2$ on the plane.

This result will be seen on one hand to be the two-dimensional analogue of the theorems given in [1, pp. 130–131] on the connection between locally absolutely continuous functions and the $L_2$-derivative in one dimension and on the other hand to be the $L_2$-analogue of the theorems obtained in [6] for pointwise convergence of the line integral analogue of the above generalized Laplace operator. It will be clear from what follows that a similar theory can be developed in the $L_1$-case and also in the $L_p$-case with $p$ between 1 and 2.

2. We shall operate in $n$-dimensional Euclidean space $E_2$, $n \geq 2$, and shall use vectorial notation. Thus,

$$
x = (x_1, \cdots, x_n), \quad y = (y_1, \cdots, y_n), \quad (x, y) = x_1y_1 + \cdots + x_ny_n, \n\alpha x + \beta y = (\alpha x_1 + \beta y_1 + \cdots + \alpha x_n + \beta y_n), \quad (x, x)^{1/2} = |x|.
$$

The open $n$-dimensional sphere with center $x_0$ and radius $r$ will be designated by $D_n(x_0, r)$. Given $F(x)$ integrable on $D_n(x_0, r)$, we shall designate the mean value of $F$ in this sphere by $A(F, x_0, r)$. Thus let-

Received by the editors March 2, 1955 and, in revised form, July 6, 1955.

1 National Science Foundation Fellow.
We further set \( \nabla(F, x_0, r) = A(F, x_0, r) - F(x_0) \) and say that \( F(x) \) has \( f(x) \) as a generalized \( L_2 \)-Laplacian, designated by \( L_2 \Delta F = f \), if
\[
\| 2(n + 2)\nabla(F, x, r)/r^2 - f(x) \|_{L^2} \to 0 \quad \text{as } r \to 0,
\]
that is if
\[
\lim_{r \to 0} \left\{ \int_{D_n(0, r)} \left| 2(n + 2)\nabla(F, x, r)/r^2 - f(x) \right|^2 dx \right\}^{1/2} = 0.
\]

\( F(x) \) will be said to be a local \( L_2 \)-potential of \( f(x) \) if for every \( r > 0 \) there is a function \( h_r(x) \), harmonic in \( D_n(0, r) \), such that
\[
F(x) = -\omega_n^{-1} \int_{D_n(0, r)} \Phi_n(x - y)f(y)dy + h_r(x)
\]
for almost all \( x \) in \( D_n(0, r) \), where \( \Phi_n(x) = |x|^{-(n-1)}(n-2)^{-1} \) if \( n \geq 3 \) and \( = \log |x|^{-1} \) if \( n = 2 \), \( \omega_n \) is the \( (n-1) \)-dimensional volume of the surface of the unit sphere, and where \( f(x) \) is in \( L_2 \) on \( E_n \).

\( G(u) \) will be said to be the Fourier transform of \( F(x) \) if
\[
G(u) = (2\pi)^{-n} \lim_{R \to \infty} \int_{D_n(0, R)} e^{-i(x, u)}F(x)dx.
\]

3. We shall prove the following theorem:

**Theorem 1.** A necessary and sufficient condition that \( F(x) \) in \( L_2 \) on \( E_n \) has \( f(x) \) as a generalized \( L_2 \)-Laplacian is that \( F(x) \) be a local \( L_2 \)-potential of \( f(x) \).

Before proving this theorem, we point out that if \( F(x) \) has \( f(x) \) as a generalized \( L_2 \)-Laplacian, then by the above theorem and \([2],[3]\), and \([4]\), \( F(x) \) has an ordinary Laplacian almost everywhere equal to \( f(x) \). That the converse of this statement does not hold can be seen in the following manner. Take \( E \) to be a closed planar set of measure zero and positive capacity contained in the interior of the disc \( D_2(0, 1/2) \). Let \( F(x) \) be the equilibrium potential of unit mass distributed on \( E \). Then \( F(x) \) is harmonic everywhere on the complement of \( E \) but not in the whole disc \( D_2(0, 1/2) \). Let \( \lambda(x) \) be the localizing function of class \( C(\infty) \) for the discs \( D_2(0, 1/2) \) and \( D_2(0, 1) \), that is, \( \lambda(x) = 1 \) for \( x \) in \( D_2(0, 1/2) \) and \( \lambda(x) = 0 \) for \( x \) in the complement of
Then \( \lambda(x)F(x) \) is in \( L_2 \) on the plane and \( \Delta [\lambda(x)F(x)] \) exists almost everywhere and is also in \( L_2 \) on the plane. But \( \Delta (\lambda F) \) is not the generalized \( L_2 \)-Laplacian of \( \lambda F \). For if it were by the theorem to be proved in this paper, \( F \) would be harmonic in the whole disc \( D_2(0, 1/2) \), which it is not.

4. In order to prove the necessary condition of the theorem, we need the following lemma:

**Lemma.** Let \( F(x) \) and \( f(x) \) be in \( L_2 \) on \( 
abla \) and suppose that \( G(u) \) and \( g(u) \) are their respective Fourier transforms. Then if for almost all \( u \), \( g(u) = -|u|^2G(u) \), \( F(x) \) is a local \( L_2 \)-potential of \( f(x) \).

To prove the lemma, set \( F_R(x) = \int e^{-|u||x-u|}G(u)du \) and \( f_R(x) = \int e^{-|u||x-u|}g(u)du \). Then \( \Delta F_R(u) = f_R(u) \), and consequently for \( x \) in \( D_n(0, T) \),

\[
F_R(x) = -\omega_n^{-1}\int_{D_n(0, T)} \Phi_n(x-y)f_R(y)dy + h_{R,T}(x)
\]

where \( h_{R,T}(x) \) is harmonic in \( D_n(0, T) \). Since \( F_R(x) \) and \( f_R(x) \) are in \( C(\infty) \) in \( \nabla \), we also have that \( h_{R,T}(x) \) is in \( L_1 \) on \( D_n(0, T) \). Next we observe that \( F_R(x) \) tends to \( F(x) \) in the \( L_1 \)-norm over \( D_n(0, T) \), and that by Fubini’s theorem and Schwarz’s inequality,

\[
\int_{D_n(0, T)} dx \int_{D_n(0, T)} |\Phi_n(x-y)| |f_R(y) - f(y)| dy \\
\leq K_T ||f_R - f||_{L_1} \rightarrow 0 \quad \text{as } R \rightarrow \infty
\]

where \( K_T \) is a constant depending on \( T \). But then from (1) we see that \( h_{R,T}(x) \) tends in \( L_1 \)-norm over \( D_n(0, T) \) to a function \( h_T(x) \) which by [5, p. 20] can be taken to be harmonic in \( D_n(0, T) \). We consequently conclude from (1) that for almost all \( x \) in \( D_n(0, T) \),

\[
F(x) = -\omega_n^{-1}\int_{D_n(0, T)} \Phi_n(x-y)f(y)dy + h_T(x),
\]

which fact proves the lemma.

5. To prove the necessary condition of the theorem, let \( G(u) \) and \( g(u) \) be the respective Fourier transforms of \( F(x) \) and \( f(x) \). Then since for fixed \( t \), \( A(F, x, t) \) is in \( L_2 \) on \( \nabla \), we have that \( \nabla(F, x, t) = A(F, x, t) - F(x) \) is in \( L_2 \) on \( \nabla \). This in conjunction with the fact that for fixed \( u \), \( A(e^{i(x,u)}, x_0, t) = 2^{n+1}\Gamma(n+2)(|u|^t - (|u|^t - 1)) J_{n+1}(|u|^t) e^{i(x_0,x)} \) where \( \mu = (n-2)/2 \) and \( J_{n+1}(t) \) is the Bessel function of the first kind.
of order \( \mu + 1 \), tells us that the Fourier transform of \( 2(n+2)\nabla(F, x, t) \) is 
\[
\eta(|u|t)G(u) \text{ where }
\eta(t) = -4(\mu + 2) \left[ 1 - 2^{\mu+1}t^{-(\mu+1)}J_{\mu+1}(t) \right].
\]

By assumption we have that \( \|2(n+2)\nabla(F, x, t)/t^2 - f(x)\|_{L^2} \to 0 \) as \( t \to 0 \); so by the Plancherel theorem, we obtain that \( \|\eta(|u|t)G(u)/t^2 - g(u)\|_{L^2} \to 0 \). Therefore there exists a sequence \( \{t_j\} \) with \( t_j \to 0 \) as \( j \to \infty \) such that for almost all \( u \), \( \eta(|u|t_j)G(u)/t_j^2 \to g(u) \). But \( \eta(|u|t_j)/t_j^2 \to -|u|^2 \), and we conclude that \( -|u|^2G(u) = g(u) \) for almost all \( u \). The lemma then tells us that \( F(x) \) is a local \( L^2 \)-potential of \( f(x) \) and the necessary part of the theorem is proved.

6. To prove the sufficiency part of the theorem, let \( t \) less than 1 be given and choose \( R \) large. Then by assumption for almost all \( x \) in \( D_n(0, R/2) \),
\[
(2) \quad F(x) = -\omega_n^{-1} \int_{D_n(0,R)} \Phi_n(x - y)f(y)dy + h_R(x)
\]
where \( h_R(x) \) is harmonic in \( D_n(0, R) \). Let \( x_0 \) be a point in \( D_n(0, R/2) \) for which \( (2) \) holds. Then
\[
\nabla(F, x_0, t) = -\omega_n^{-1} \int_{D_n(0,R)} f(y)\nabla(\Phi_n(x - y), x_0, t)dy.
\]
But from the fact that for fixed \( y \),
\[
A(\Phi_n(x - y), x_0, t) = \Phi_n(x_0 - y) \quad \text{ if } |x_0 - y| > t
\]
\[
= nt^{n-2}\left[t^n(n - 2)^{-1} - |x_0 - y|^{2n-1}\right] \quad \text{ if } |x_0 - y| \leq t \text{ and } n \geq 3
\]
\[
= 2^{-1}[1 - 2 \log t - |x_0 - y|^{2t-2}] \quad \text{ if } |x_0 - y| \leq t \text{ and } n = 2,
\]
we obtain that
\[
\nabla(F, x_0, t) = -\omega_n^{-1} \int_{D_n(x_0,t)} f(y)\nabla(\Phi_n(x - y), x_0, t)dy.
\]
We conclude that for almost all \( x \) in \( E_n \),
\[
[2(n+2)\nabla(F, x, t)/t^2 - f(x)]
\]
\[
= -\omega_n^{-1}2(n + 2) \int_{D_n(0,t)} [f(x + y) - f(x)]\psi_n(t, y)dy
\]
where
\[ \psi_n(t, y) = \begin{cases} nt^{-n}2^{-1}[t^2(n - 2)^{-1} - |y|^{2n-1}] - |y|^{2-n}(n - 2)^{-1}/t^2 & \text{if } n \geq 3, \\ 2^{-1}[1 - |y|^{2t-2}] - \log t |y|^{-1}/t^2 & \text{if } n = 2, \end{cases} \]

and

\[
\int_{D_n(0, t)} \psi_n(t, y) dy = -\omega_n [2(n + 2)]^{-1}.
\]

From (3) and (4) and the Schwarz inequality, we obtain that for almost all \( x \) in \( E_n \),

\[
2(n + 2) \nabla(F, x, t)/t^2 - f(x) \leq K'_n \max_{0 < |y| \leq t} \| f(x + y) - f(x) \|_{L_1} \leq K'_n \int_{D_n(0, t)} |f(x + y) - f(x)|^2 \psi_n(t, y) dy
\]

where \( K'_n \) is another constant independent of \( t \). From (4), (5), and Fubini's theorem, we conclude that

\[
\|2(n + 2) \nabla(F, x, t)/t^2 - f(x)\|_{L_2} \leq K'_n \max_{0 < |y| \leq t} \| f(x + y) - f(x) \|_{L_1} = o(1) \text{ as } t \to 0,
\]

which fact is the desired result.

7. Extending the definition for the generalized \( L_p \)-Laplacian and local \( L_p \)-potential in an obvious manner to \( L_p \)-spaces for \( 1 \leq p \leq 2 \), using the Hölder inequality or obvious facts about functions in \( L_1 \) where we have previously used the Schwartz inequality and using the Titchmarsh-Hausdorff-Young theorem where we have previously used the Plancherel theorem, we obtain the following extension of Theorem 1.

**Theorem 2.** Let \( 1 \leq p \leq 2 \). Then a necessary and sufficient condition that \( F(x) \) in \( L_p \) on \( E_n \) has \( f(x) \) as a generalized \( L_p \)-Laplacian is that \( F(x) \) be a local \( L_p \)-potential of \( f(x) \).

**References**


Rutgers University and
The Institute for Advanced Study