GENERALIZED $L_2$-LAPLACIANS

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1. Given a function $F(x, y)$ in $\mathbb{R}^2$ on the plane, we shall say that $F(x, y)$ has $f(x, y)$ as a generalized $L_2$-Laplacian if

$$\frac{1}{t^2} \left[ \frac{1}{2\pi} \int_{D_2(0,1)} F(x + tu, y + tv) dudv - F(x, y) \right] \rightarrow f(x, y) \text{ in the } L_2\text{-norm}$$

where $D_2(0, 1)$ is the disc with radius 1 and center at the origin. It will be shown here that a necessary and sufficient condition that $F(x, y)$ has $f(x, y)$ as a generalized $L_2$-Laplacian is that $F(x, y)$ be locally an $L_2$-potential of $f(x, y)$, that is given any disc $D_2(0, R)$ there is a harmonic function $h_R(x, y)$ such that for almost all $(x, y)$ in $D_2(0, R)$

$$F(x, y) = -\left(2\pi\right)^{-1} \int_{D_2(0, R)} \log \left[ (x - u)^2 + (y - v)^2 \right]^{-1/2} f(u, v) dudv$$

$$+ h_R(x, y)$$

where $f(x, y)$ is in $L_2$ on the plane.

This result will be seen on one hand to be the two-dimensional analogue of the theorems given in [1, pp. 130–131] on the connection between locally absolutely continuous functions and the $L_2$-derivative in one dimension and on the other hand to be the $L_2$-analogue of the theorems obtained in [6] for pointwise convergence of the line integral analogue of the above generalized Laplace operator. It will be clear from what follows that a similar theory can be developed in the $L_1$-case and also in the $L_p$-case with $p$ between 1 and 2.

2. We shall operate in $n$-dimensional Euclidean space $E_n$, $n \geq 2$, and shall use vectorial notation. Thus,

$$x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n), \quad (x, y) = x_1y_1 + \cdots + x_ny_n,$$

$$\alpha x + \beta y = (\alpha x_1 + \beta y_1 + \cdots + \alpha x_n + \beta y_n), \quad (x, x)^{1/2} = |x|.$$

The open $n$-dimensional sphere with center $x_0$ and radius $r$ will be designated by $D_n(x_0, r)$. Given $F(x)$ integrable on $D_n(x_0, r)$, we shall designate the mean value of $F$ in this sphere by $A(F, x_0, r)$. Thus let-

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We further set $\nabla(F, x_0, r) = A(F, x_0, r) - F(x_0)$ and say that $F(x)$ has $f(x)$ as a generalized $L_2$-Laplacian, designated by $L_2\Delta F = f$, if

$$\|2(n + 2)\nabla(F, x, r)/r^2 - f(x)\|_{L^2} \to 0 \quad \text{as} \quad r \to 0,$$

that is if

$$\lim_{r \to 0} \left\{ \int_{E_n} \left| 2(n + 2)\nabla(F, x, r)/r^2 - f(x) \right|^2 \, dx \right\}^{1/2} = 0.$$

$F(x)$ will be said to be a local $L_2$-potential of $f(x)$ if for every $r > 0$ there is a function $h_r(x)$, harmonic in $D_n(0, r)$, such that

$$F(x) = -\omega_n^{-1} \int_{D_n(0, r)} \Phi_n(x - y)f(y) \, dy + h_r(x),$$

for almost all $x$ in $D_n(0, r)$, where $\Phi_n(x) = |x|^{-(n-2)}(n-2)^{-1}$ if $n \geq 3$ and $=\log |x|^{-1}$ if $n = 2$, $\omega_n$ is the $(n-1)$-dimensional volume of the surface of the unit sphere, and where $f(x)$ is in $L_2$ on $E_n$.

$G(u)$ will be said to be the Fourier transform of $F(x)$ if

$$G(u) = (2\pi)^{-n} \lim_{R \to \infty} \int_{D_n(0, R)} e^{-i(x \cdot u)}F(x) \, dx.$$

3. We shall prove the following theorem:

**Theorem 1.** A necessary and sufficient condition that $F(x)$ in $L_2$ on $E_n$ has $f(x)$ as a generalized $L_2$-Laplacian is that $F(x)$ be a local $L_2$-potential of $f(x)$. Before proving this theorem, we point out that if $F(x)$ has $f(x)$ as a generalized $L_2$-Laplacian, then by the above theorem and [2], [3], and [4], $F(x)$ has an ordinary Laplacian almost everywhere equal to $f(x)$. That the converse of this statement does not hold can be seen in the following manner. Take $E$ to be a closed planar set of measure zero and positive capacity contained in the interior of the disc $D_4(0, 1/2)$. Let $F(x)$ be the equilibrium potential of unit mass distributed on $E$. Then $F(x)$ is harmonic everywhere on the complement of $E$ but not in the whole disc $D_4(0, 1/2)$. Let $\lambda(x)$ be the localizing function of class $C(\infty)$ for the discs $D_2(0, 1/2)$ and $D_4(0, 1)$, that is, $\lambda(x) = 1$ for $x$ in $D_2(0, 1/2)$ and $\lambda(x) = 0$ for $x$ in the complement of
$D_2(0, 1)$. Then $\lambda(x) F(x)$ is in $L_2$ on the plane and $\Delta [\lambda(x) F(x)]$ exists almost everywhere and is also in $L_2$ on the plane. But $\Delta (\lambda F)$ is not the generalized $L_2$-Laplacian of $\lambda F$. For if it were by the theorem to be proved in this paper, $F$ would be harmonic in the whole disc $D_2(0, 1/2)$, which it is not.

4. In order to prove the necessary condition of the theorem, we need the following lemma:

**Lemma.** Let $F(x)$ and $f(x)$ be in $L_2$ on $E_n$ and suppose that $G(u)$ and $g(u)$ are their respective Fourier transforms. Then if for almost all $u$, $g(u) = -|u|^2 G(u)$, $F(x)$ is a local $L_2$-potential of $f(x)$.

To prove the lemma, set $F_R(x) = \int_{E_n} e^{-i|x-u|} G(u) du$ and $f_R(x) = \int_{E_n} e^{-i|x-u|} g(u) du$. Then $\Delta F_R(u) = f_R(u)$, and consequently for $x$ in $D_n(0, T)$,

$$F_R(x) = -\omega_n^{-1} \int_{D_n(0,T)} \Phi_n(x - y) f_R(y) dy + h_{R,T}(x)$$

where $h_{R,T}(x)$ is harmonic in $D_n(0, T)$. Since $F_R(x)$ and $f_R(x)$ are in $C(\infty)$ in $E_n$, we also have that $h_{R,T}(x)$ is in $L_1$ on $D_n(0, T)$. Next we observe that $F_R(x)$ tends to $F(x)$ in the $L_1$-norm over $D_n(0, T)$, and that by Fubini’s theorem and Schwarz’s inequality,

$$\int_{D_n(0,T)} dx \int_{D_n(0,T)} |\Phi_n(x - y)| |f_R(y) - f(y)| dy \leq K_T ||f_R - f||_{L_1} \rightarrow 0$$

as $R \rightarrow \infty$ where $K_T$ is a constant depending on $T$. But then from (1) we see that $h_{R,T}(x)$ tends in $L_1$-norm over $D_n(0, T)$ to a function $h_T(x)$ which by [8, p. 20] can be taken to be harmonic in $D_n(0, T)$. We consequently conclude from (1) that for almost all $x$ in $D_n(0, T)$,

$$F(x) = -\omega_n^{-1} \int_{D_n(0,T)} \Phi_n(x - y) f(y) dy + h_T(x),$$

which fact proves the lemma.

5. To prove the necessary condition of the theorem, let $G(u)$ and $g(u)$ be the respective Fourier transforms of $F(x)$ and $f(x)$. Then since for fixed $t$, $A(F, x, t)$ is in $L_2$ on $E_n$, we have that $\nabla (F, x, t) = A(F, x, t) - F(x)$ is in $L_2$ on $E_n$. This in conjunction with the fact that for fixed $u$, $A(e^{i(x,u)}, x_0, t) = 2^{n+1} \Gamma(\mu + 2)(|u|t)^{-\mu+1} J_{\mu+1}(\mu t) e^{i(x_0,u)}$

where $\mu = (n-2)/2$ and $J_{\mu+1}(t)$ is the Bessel function of the first kind
of order $\mu + 1$, tells us that the Fourier transform of $2(n + 2)\nabla(F, x, t)$ is $\eta(|u| t)G(u)$ where

$$\eta(t) = -4(\mu + 2)\left[1 - 2^{\mu+1}\Gamma(\mu + 2)t^{-(\mu + 1)}J_{\mu+1}(t)\right].$$

By assumption we have that $\|2(n + 2)\nabla(F, x, t)/t^2 - f(x)\|_{L^2} \to 0$ as $t \to 0$; so by the Plancherel theorem, we obtain that $\|\eta(|u| t)G(u)/t^2 - g(u)\|_{L^2} \to 0$. Therefore there exists a sequence $\{t_j\}$ with $t_j \to 0$ as $j \to \infty$ such that for almost all $u$, $\eta(|u| t_j)G(u)/t_j^2 \to g(u)$. But $\eta(|u| t_j)/t_j^2 \to -|u|^2$, and we conclude that $-|u|^2G(u) = g(u)$ for almost all $u$. The lemma then tells us that $F(x)$ is a local $L^2$-potential of $f(x)$ and the necessary part of the theorem is proved.

6. To prove the sufficiency part of the theorem, let $t$ less than 1 be given and choose $R$ large. Then by assumption for almost all $x$ in $D_n(0, R/2)$,

$$(2) \quad F(x) = -\omega_n^{-1}\int_{D_n(0, R)} \Phi_n(x - y)f(y)dy + h_B(x)$$

where $h_B(x)$ is harmonic in $D_n(0, R)$. Let $x_0$ be a point in $D_n(0, R/2)$ for which (2) holds. Then

$$\nabla(F, x_0, t) = -\omega_n^{-1}\int_{D_n(0, R)} f(y)\nabla(\Phi_n(x - y), x_0, t)dy.$$ 

But from the fact that for fixed $y$,

$$A(\Phi_n(x - y), x_0, t) = \Phi_n(x_0 - y) \quad \text{if } |x_0 - y| > t$$

$$= nt^{n-2}[t^{n-2}(n - 2)^{-1} - |x_0 - y|^{2n-1}] \quad \text{if } |x_0 - y| \leq t \text{ and } n \geq 3$$

$$= 2^{-1}[1 - 2 \log t - |x_0 - y|^{2t-2}] \quad \text{if } |x_0 - y| \leq t \text{ and } n = 2,$$

we obtain that

$$\nabla(F, x_0, t) = -\omega_n^{-1}\int_{D_n(x_0, t)} f(y)\nabla(\Phi_n(x - y), x_0, t)dy.$$ 

We conclude that for almost all $x$ in $E_n$,

$$[2(n + 2)\nabla(F, x, t)/t^2 - f(x)]$$

$$= -\omega_n^{-1}2(n + 2)\int_{D_n(0, t)} [f(x + y) - f(x)]\psi_n(t, y)dy$$

where
\[ \psi_n(t, y) = \begin{cases} \frac{n^{n-2} [t^2(n - 2)^{-1} - |y|^2 n^{-1}] - |y|^{2n(n - 2)^{-1}}}{t^2} & \text{if } n \geq 3, \\ 2^{-1} [1 - |y|^{2n-2}] - \log t |y|^{-1} / t^2 & \text{if } n = 2, \end{cases} \]

and

\[ \int_{D_n(0, t)} \psi_n(t, y) \, dy = -\omega_n [2(n + 2)]^{-1} \]

Using the Holder inequality or obvious facts about functions in \( L^p \) where we have previously used the Schwartz inequality and using the Titchmarsh-Hausdorff-Young theorem where we have previously used the Plancherel theorem, we obtain the following extension of Theorem 1.

**Theorem 2.** Let \( 1 \leq p \leq 2 \). Then a necessary and sufficient condition that \( F(x) \) in \( L^p \) on \( \mathbb{R}^n \) has \( f(x) \) as a generalized \( L^p \)-Laplacian is that \( F(x) \) be a local \( L^p \)-potential of \( f(x) \).

**References**


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