ON A CLASS OF SEMIGROUPS ON E_n

PAUL S. MOSTERT AND ALLEN L. SHIELDS[1]

Very little is known about continuous, associative multiplications with identity on a manifold unless the manifold is compact (in which case it must be a Lie group [1]). In this, and in forthcoming papers, the authors will study this problem, particularly when the manifold is E_n (n-dimensional Euclidean space), and with other restrictions. Here, we classify all (topological) semigroups on the half line $[0, \infty)$ in which 0 and 1 play their natural roles of zero and identity, and use this as an aid in classifying semigroups S with identity on E_n, $n>1$, such that some $(n-1)$-dimensional compact connected submanifold is a subsemigroup containing the identity of S. It turns out that only E_2 and E_4 will admit such a situation. In fact, we prove the following two theorems (see §1 for definitions):

THEOREM A. *Let S be the half-line $[0, \infty)$. Suppose S is a semigroup with zero at* 0 *and identity at* 1. *Then*

(i) *if S contains no other idempotents, its multiplication is the ordinary multiplication of real numbers on $[0, \infty)$;*

(ii) *if S contains an idempotent different from* 0 *and* 1, *then it contains a largest (in the sense of the regular order of real numbers) such idempotent e. Moreover, $e<1$, $[e, \infty)$ is a subsemigroup iseomorphic to $[0, \infty)$ under the usual multiplication of real numbers, and $[0, e]$ is an (I)-semigroup.*

THEOREM B. *Let S be a semigroup with identity on E_n, $n>1$, and B a compact, connected submanifold of dimension $n-1$. If B is a subsemigroup containing the identity of S, then:*

(i) *$n=2$ or 4 and B is a Lie group which is S^1 if $n=2$ and S^3 if $n=4$ (where S^i denotes the i-sphere);*

(ii) *there exists a subsemigroup J contained in the center of S which is iseomorphic to a semigroup of the type described in Theorem* A;

(iii) *the subsemigroup J meets each orbit $xB=Bx$ of B in exactly one point, and $JB=S$;*

(iv) *if* 0 *denotes the zero of J, then* 0 *is a zero for S, and $(J\backslash\{0\})\times B$ is iseomorphic to $(J\backslash\{0\})B=S\backslash\{0\}$ in the natural way.*

Presented to the Society, November 19, 1955; received by the editors August 30, 1955 and, in revised form, November 26, 1955.

[1] This work was done during the summer of 1955 while the authors were at the Institute for Advanced Study. They received their support from Tulane University's National Science Foundation contract.

1. **Preliminaries.** We shall use the word *semigroup* to mean *topological semigroup*, that is, a Hausdorff space with a continuous, associative multiplication. A *subsemigroup* will always be closed. An *ideal* of the semigroup S is a set $A \subset S$ such that $SAS \subset A$. A *zero* for S is an element 0 such that $0x = x0 = 0$ for all $x \in S$.

In place of the phrase "simultaneous isomorphism and homeomorphism" we use the word *iseomorphism*. The symbol $\square$ will be used to denote the null set, and $A \backslash B$ to denote the set-theoretic difference of the sets A and B.

If S is a semigroup with identity 1, we denote by $H(1)$ the set of all elements with two-sided inverses relative to 1. We shall refer to $H(1)$ as the *maximal subgroup* of the identity. In general, $H(1)$ need not be a topological group since inversion is not necessarily continuous.

Let S be a semigroup, and for $x \in S$, let $\Gamma(x) = \{x, x^2, \cdots\}^-$. If $\Gamma(x)$ is compact, it contains exactly one idempotent f. If $fx = x$, then $\Gamma(x)$ is a group [1].

An (I)-*semigroup* is a semigroup on a closed interval of the line such that one end point functions as a zero and the other as an identity for the semigroup. Hence, if $[0, e]$ is an (I)-semigroup with 0 a zero, then $ex = xe = x$ for all $x \in [0, e]$. The complete structure of (I)-semigroups (and also of (L)-semigroups as described below) has been obtained by the authors [2].

An (L)-*semigroup* is a semigroup S with identity on a compact manifold with connected boundary B such that B is a Lie group. There is an (I)-semigroup J contained in the center of S such that $JB = S$, and such that J meets each orbit $xB = Bx$ of B in exactly one point. If S is the two or four cell, and B is S^1 or S^3 respectively, then the single exceptional orbit of B is a point which acts as a zero for S. Obviously, this description implies S/B is an (I)-semigroup iseomorphic under the natural projection to J.

2. **Proof of Theorem A.**

Lemma. *If S is a semigroup with identity such that $Q = S \backslash H(1)$ has compact closure, then Q is an ideal.*

Proof. Since $H(1)$ is a group, if $x \in Q$, $y \in H(1)$, then quite trivially $xy \notin H(1)$. Hence, we may assume $y \in Q$. Now if $xy = g \in H(1)$, then $z = yg^{-1} \in Q$ and $xz = 1$. Multiplying successively on the left by x and on the right by z, we have $x^n z^n = 1$. Suppose $x^n \in H(1)$ for some n, say $x^n = h$. Then $x(x^{n-1}h^{-1}) = (h^{-1}x^{n-1})x = 1$, so that $x \in H(1)$. Hence, $\Gamma(x)$ and $\Gamma(z)$ are compact, and we may assume, for some subsequence, $x^{n(i)} \to f$, $z^{n(i)} \to w$, where f is idempotent. Hence, $fw = 1$. Multiplying on the left by f, we have $f^2 w = f$, and since $f^2 = f$, $f = 1$. This implies

$\Gamma(x)$ is a group, and $x \in H(1)$, a contradiction. Thus, $QS \cup SQ \subset Q$.

We now are prepared to prove the theorem. For any $\epsilon > 0$, let I_ϵ be a closed interval about 1 of radius ϵ. Let a and b denote the end points of I_ϵ. By the continuity of multiplication (which is uniform on compact sets) there exists an interval $I_\delta \subset I_\epsilon$ of radius $\delta > 0$, $1 \in I_\delta$, such that $|ax-a|$, $|xa-a|$, $|bx-b|$, $|xb-b|$, are all less than ϵ for $x \in I_\delta$. Hence, for $x \in I_\delta$, $1 \in xI_\epsilon \cap I_\epsilon x$; that is, x has a right inverse and a left inverse in I_ϵ. Since the two, if they exist, must coincide, x has an inverse which we denote by x^{-1}. Clearly $x \rightarrow x^{-1}$ is continuous (i.e., $x^{-1} \rightarrow 1$ as $x \rightarrow 1$).

Now choose a connected neighborhood V of 1 such that $V = V^{-1}$ (since I_ϵ is a locally compact local group, this can always be done). Let $G = \bigcup_n V^n$. Then G is a one-dimensional connected Lie group, and hence is either the circle or the line. Since $G \subset E_1$, G is the line. Clearly, G^- is either of the form $[e, f]$ or $[e, \infty)$. If it is $[e, f]$, then, since $x^n \rightarrow e$ implies $x^{-n} \rightarrow f$, $ef = fe = 1$. In this case, $[e, f]$ is a compact group, which is impossible. Hence, $G^- = [e, \infty)$, and $G = (e, \infty)$. Now if $x \in G$, and $x < 1$, then $x^n \rightarrow e$. It follows that $e^2 = e$. Obviously, e is the largest idempotent different from 1, and $ex = xe = e$ for $x \in G$. This proves that $[e, \infty)$ is iseomorphic to $[0, \infty)$ under the usual multiplication of real numbers.

Clearly G is an open subgroup of $H(1)$ so that, trivially, $H(1)$ is a topological group. Suppose $x \in H(1) \backslash G$. Then $xG \subset [0, e]$, so that $(xG)^-$ is a closed interval, say with end points g and f. Then for some $y \in G$, $xy^n \rightarrow g$, $xy^{-n} \rightarrow f$. Since $x \in H(1)$, $y^n \rightarrow x^{-1}g$, $y^{-n} \rightarrow x^{-1}f$. But we have already seen that either $y^n \rightarrow \infty$ or $y^{-n} \rightarrow \infty$. Hence, $H(1) = G$.

By the lemma, $[0, e] = S \backslash H(1)$ is an ideal. Moreover, since $e[0, e]$, and $[0, e]e$ are connected sets containing 0 and e, e acts as an identity for $[0, e]$ so that $[0, e]$ is an (I)-semigroup. This completes the proof of Theorem A.

Notice that if $x \in [0, e]$, $y \in [e, \infty)$, then $xy = yx = x$ since e acts as a zero for $[e, \infty)$ and an identity for $[0, e]$.

3. **Proof of Theorem B.** Since B is a compact manifold which is a semigroup with identity, it is a Lie group [1]. By a theorem of Montgomery and Zippin [3], B is an $(n-1)$-sphere, and hence is S^1 or S^3. Thus, $n = 2$ or 4. Moreover, all orbits, except one, are homeomorphic to B (i.e., B is simply transitive on each orbit), and the exceptional orbit is a point (where we can assume B operates on the left or right of S by $g(x) = gx$ or $g(x) = xg$ respectively).

Let X denote the space of right orbits, xB, with the usual identification topology where π: $S \rightarrow X$ is the natural projection. It is also proved in [3] that X has a cross section C so that $CB = S$. It follows that X is homeomorphic to $[0, \infty)$, the half-line. We shall identify X with $[0, \infty)$ where $\pi(1)=1$ and $\pi(x_0)=0$, where x_0 denotes the point in the exceptional right orbit. The proof proceeds in several short steps which we shall number for easy reference.

3.1. *If $n(xB, x_0)$ denotes the index (degree) of the mapping f_x: $B \rightarrow S$ defined by $f_x(b)=xb$ relative to x_0 then $n(xB, x_0)=1$ for $x \neq x_0$.*

Proof. Let J be a cross section from x to 1 of $[\pi(x), \pi(1)]$, (see [**3**]). Then $x_0 \notin JB$ so that $n(jB, x_0)$ is defined for all $j \in J$ and is constant. But $n(B, x_0)=1$.

3.2. *Define $x<y$ if $\pi(x)<\pi(y)$. Then $x<y$ if and only if $n(yB, x)=1$.*

Proof. Suppose $x<y$. Then there exists a cross section J of $[0, \pi(x)]$ from x_0 to x, $y \notin JB$. Hence, $n(yB, x)=n(yB, x_0)=1$. On the other hand, if $n(yB, x)=1$, then clearly $\pi(x) \neq \pi(y)$, for otherwise $x \in yB$ so that $n(yB, x)$ is not defined. Now if $x>y$, then $n(yB, x) =n(x_0B, x)=0$. Hence, the only alternative is $x<y$.

3.3. *If $x<y$, then $bx<by$, and $xb<yb$, $b \in B$.*

Proof. The second statement is trivial. Now suppose $b \in B$ and $bx \in byB$. If we left multiply by b^{-1}, we have $x \in yB$. Hence, $bx \notin byB$ for $b \in B$, and $n(byB, bx)$ is defined. Since B is connected, and $gx \notin gyB$ for every $g \in B$, $n(byB, bx)=n(yB, x)=1$ which implies the result by 3.2.

3.4. *For every $x \in S$, $xB=Bx$. Moreover, X is a semigroup on $[0, \infty)$ with zero at 0 and identity at 1.*

Proof. We shall show that $bx>x$ and $x>bx$ are both false. This will then tell us that $bx \in xB$. The dual argument will show $xb \in Bx$.

Assume $x>bx$. Left multiply by b^{-1} to get, by 3.3,

$$b^{-1}x > x > bx.$$

Left multiply by b successively to get

$$b^{-1}x > x > bx > b^2x > \cdots > b^nx > \cdots$$

for all positive integers n. Since B is a compact group, $\Gamma(b)$ is a group with the property that

$$x > cx, \qquad c \in \Gamma(b).$$

But $b^{-1} \in \Gamma(b)$ so that

$$b^{-1}x > x > b^{-1}x$$

which is clearly impossible.

Similarly, $x<bx$ is impossible.

Clearly, then, X is a semigroup with identity $1=\pi(B)$. Moreover, since $x_0B=x_0$, $xx_0B=xx_0$, so that xx_0 is a singular orbit. Since there is just one, $xx_0=x_0$ for every x. Since $x_0B=Bx_0$, x_0 is also the unique singular left orbit so that similarly $x_0x=x_0$ for $x\in S$. That is, x_0 is a zero for S. It follows that $\pi(x_0)=0$ is a zero for X.

3.5. *Let $D=\pi^{-1}([0, 1])$. Then D and $Q=(S\backslash D)^-$ are subsemigroups.*

PROOF. If x, $y\in D$, then $\pi(x)\pi(y)=\pi(y)\pi(x)\in[0, 1]$. Hence, since $xy\in\pi^{-1}(\pi(xy))$, $xy\in D$. Thus, D is a subsemigroup.

Now if x, $y\in Q$, a similar argument shows $xy\in Q$. (See Theorem A.)

We now complete the proof of the theorem. Let T be an (I)-semigroup in D which maps iseomorphically under π on $[0, 1]$ and such that T is in the center of D. This can be done since obviously D is an (L)-semigroup on the two-cell or four-cell as the case may be. Let e denote the maximal idempotent of T different from 1 (which exists by Theorem A). Since T is iseomorphic to $[0, 1]\subset[0, \infty)$, $[\pi(e), 1]$ is iseomorphic to $[0, 1]$ under the usual multiplication of real numbers (Theorem A), and every element in $(\pi(e), 1]$ has an inverse in $[1, \infty)$. Let T_0 denote that portion of T from e to 1, excluding e (i.e., T_0 is iseomorphic to $(\pi(e), 1])$. For each $x\in T_0$, there exists an element $y\in Q$ such that $xy\in B$, since xB has an inverse in X. Say $xy=b$. Let $x^{-1}=yb^{-1}$. Clearly, using a like argument on the other side, x has a left and a right inverse and hence a unique inverse, and this is x^{-1}. Obviously $x\rightarrow x^{-1}$ is one-one. Moreover, if $x_n\rightarrow 1$, then, for some subsequence, $x_n^{-1}\rightarrow b\in B$, for some b. Then $x_nx_n^{-1}\rightarrow 1b$ so that $b=1$. Hence $x\rightarrow x^{-1}$ is continuous, and similarly, $x^{-1}\rightarrow x$ is continuous (in fact, they are iseomorphisms). Let σ: $T_0\rightarrow S$ be the function defined by $\sigma(x)=x^{-1}$, and $T'=\sigma(T_0)$. Then, $\pi(T')=[1, \infty)$.

Since T_0 and T' are one-parameter semigroups, for $x\in T_0$, $y\in T'$, either there is an element $z\in T'$ such that $y=x^{-1}z$, or there exists an element $z\in T_0$ such that $x=y^{-1}z$. Then $xy=z\in T'\cup T_0$. Moreover, e being a zero for T_0, is clearly a zero for T' also. It follows that $J=T\cup T'$ is closed under multiplication and is a cross section of X. Moreover, since $xb=bx$ for $x\in T$, $b\in B$, clearly the same is true for all $x\in J$.

Now if $x\in J$, $y\in S$, $y=x'b'$ for some $x'\in J$, $b'\in B$. Hence,

$$xy = xx'b' = x'b'x$$

since J is abelian and commutes elementwise with B. Thus, J is in the center of S. The remainder of the theorem is immediate.

4. **Other semigroups on** E_1. If we assume that S is a semigroup on $E_1=(-\infty, \infty)$ with -1, 0, 1 playing their natural roles, then the conclusion of Theorem B follows in a similar way, where now the group $\mathrm{B}=\{1, -1\}$.

On the other hand, if we continue to assume B connected, then $B=\{1\}$. In this case, S need not be of this form, and in fact $(-\infty, 0]$ need not contain any elements of the maximal subgroup as the following example shows.

4.1. EXAMPLE. For a fixed e, $0<e<1$, take $S=(-e, \infty)$ and define multiplication $x \circ y$ in S as follows:

(i) for $x, y \in [0, e]$, $x \circ y = \min (x, y)$,

(ii) for $x, y \in (-e, 0]$, $x \circ y = |x| \circ |y|$,

(iii) for $x \in [0, e]$, $y \in (-e, 0]$, $x \circ y = y \circ x = -(|x| \circ |y|)$,

(iv) for $x, y \in [e, \infty)$, $x = z_1 + e$, $y = z_2 + e$, $x \circ y = z_1 z_2 + e$,

(v) for $x \in [e, \infty)$, $y \in (-e, e]$, $x \circ y = y \circ x = y$.

Then S is a semigroup on E_1 with a connected, compact subroup of dimension one less, namely $\{1\}$, but no element in $(-e, 0]$ has an inverse since $x \circ y \in (-e, e]$ for $x \in (-e, 0]$, $y \in S$.

BIBLIOGRAPHY

1. A. D. Wallace, *The structure of topological semigroups*, Bull. Amer. Math. Soc. vol. 61 (1955) pp. 95–112.

2. P. S. Mostert and A. L. Shields, *On the structure of semigroups on a compact manifold with boundary*, to appear in Ann. of Math.

3. D. Montgomery and L. Zippin, *A class of transformation groups in E_n*, Amer. J. Math. vol. 65 (1943) pp. 601–608.

TULANE UNIVERSITY