ON THE SPACE OF INTEGRAL FUNCTIONS. IV

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1. **Introduction.** We first recall the definitions, notations and some of the results in the previous papers [**1**; **2**; **3**].[1] We denoted by Γ the class of integral functions $\alpha=\alpha(z)=\sum_0^\infty a_n z^n$, topologised by the metric $|\alpha-\beta|$ where

$$|\alpha| = \max(|a_0|, |a_n|^{1/n}, n \geqq 1).^2$$

With this metric, Γ is a complete linear metric space and convergence of a sequence in Γ is equivalent to uniform convergence of the sequence of integral functions in any circle of finite radius [**1**, Theorems 1 and 3]. Being a locally convex space in the concerned metric, the Hahn-Banach extension theorem for continuous linear functionals remain valid (for a direct proof see [**2**, Theorems 3 and 4]). The topology of Γ can also be specified by the family of norms

$$|\alpha; R| = \sum |a_n| R^n, \qquad R > 0.$$

(See [**2**, Theorem 1].) Using this result, we proved [**3**, Theorem 6] the following criterion for the existence of a continuous linear transformation of Γ into Γ:

THEOREM 1. *Let $\delta_n=z^n$, $n=0, 1, 2, \cdots$. A necessary and sufficient condition that there exists a continuous linear transformation T of Γ into itself such that $T(\delta_n)=\alpha_n$ is that, for each $R>0$, the sequence $|\alpha_n; R|^{1/n}$ should be bounded.*

1.1. In this paper we first obtain the general forms of isometric and metrically bounded transformations (§§2 and 3). We next consider the homomorphisms of Γ into itself when it is endowed with the structure of an algebra in two ways, one when multiplication is defined as the natural product of two integral functions and the other in which multiplication is defined by Hadamard composition. Both lead to topological algebras, the first having an identity, namely δ_0, and the second having no identity. The group of automorphisms of these algebras are obtained (§§4 and 5).

2. **Isometric transformation.** In this section we consider linear isometric transformations of Γ into itself. Such a transformation T is

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[1] The numbers in square brackets refer to bibliography at the end.

[2] The symbol $|\cdot|$ is used to denote the metric in Γ as well as the modulus of complex numbers. The context will show which is meant.

isometric if and only if $|T(\alpha)| = |\alpha|$ for every $\alpha \in \Gamma$. Obviously such transformations are continuous. The following theorm gives the general form of such transformations.

THEOREM 2. *Every isometric linear transformation T of Γ into itself is of one or other of the following two types*:

Type I. $T(\delta_n) = k_n\delta_n$, $n \geqq 0$, *and* $T(\alpha) = \sum_0^\infty k_n a_n \delta_n$;

Type II. $T(\delta_0) = k_0\delta_1$, $T(\delta_1) = k_1\delta_0$, $T(\delta_n) = k_n\delta_n$, $n \geqq 2$ *and* $T(\alpha) = k_0 a_0 \delta_1 + k_1 a_1 \delta_0 + \sum_2^\infty k_n a_n \delta_n$.

In both cases, k_n, $n \geqq 0$, *are complex numbers with* $|k_n| = 1$ *and* $\alpha = \sum_0^\infty a_n \delta_n$.

Conversely a transformation of the type I *or* II *is a linear isometric transformation of* Γ *into itself.*

PROOF. The converse is trivial and in the direct part of the theorem we need prove only the expressions for $T(\delta_n)$, the remaining following from the continuity of isometric transformations. We consider two cases:

Case 1. Let $n \geqq 1$. For any complex number c we have

$$T(c^n\delta_n) = c^n T(\delta_n).$$

Writing $T(\delta_n) = \sum_{p=0}^\infty a_{np}\delta_p$, we see, by the isometry of T, that $|c^n a_{n0}| \leqq |c|$ and $|c^{n/p}| \, |a_{np}|^{1/p} \leqq |c|$ $(p \geqq 1)$. Now c is at our disposal. So letting $c \to 0$ or ∞ according as $p > n$ or $p < n$ we conclude that $a_{np} = 0$ for $p \neq n$ when $n \geqq 2$ and also when $p \neq 0$ or 1 when $n = 1$. In view of the isometry we must have $|a_{nn}| = 1$ for all $n \neq 0$ or 1.

Case 2. Let $n = 0$. An argument similar to the above shows that $a_{n0} = 0$ when $n \geqq 2$. So we have to consider the equations

$$T(\delta_0) = a_{00}\delta_0 + a_{01}\delta_1, \qquad T(\delta_1) = a_{10}\delta_0 + a_{11}\delta_1.$$

We must have max $(|a_{00}|, |a_{01}|) = \text{Max}\,(|a_{10}|, |a_{11}|) = 1$. Multiplying by k and adding we get the inequality

$$|a_{00} + ka_{10}| \leqq \max(1, |k|).$$

From this we see that if one of $|a_{00}|$, $|a_{10}|$ be equal to 1 the other is zero since otherwise, by choosing k suitably, we easily get a contradiction of the above inequality. Similarly, we can prove that if one of $|a_{01}|$, $|a_{11}|$ be equal to 1 the other is zero. Combining these conclusions we see that the results are proved.

2.1. The above theorem along with Theorem 1 gives the following result.

THEOREM 3. *Every isometric linear transformation of* Γ *into itself is*

a one-to-one bi-continuous transformation of Γ *onto itself. In other words, every isometry is also an automorphism of* Γ.

2.2. REMARK. The situation described in the previous theorem may be contrasted with what is known in many normed linear spaces. For instance, in the space of bounded sequences $x=(x_1, x_2, x_3, \cdots)$ with norm $=\max(|x_n|, n\geqq 1)$, the transformation $T(x)=x'=(0, x_1, x_2, x_3, \cdots)$ is a linear isometric transformation of the space into *a proper subspace of itself.*

3. **Metrically bounded transformations.** In this section we consider linear transformations T of Γ into itself such that $|T(\alpha)| \leqq K|\alpha|$ for all $\alpha\in\Gamma$, where K is a suitable positive constant. Such transformations are again obviously continuous and include all isometries of Γ. We prove the following result for such transformations.

THEOREM 4. *If T is a linear transformation such that $|T(\alpha)| \leqq K|\alpha|$ for all $\alpha\in\Gamma$ then*

$$T(\delta_0) = a_{00}\delta_0 + a_{01}\delta_1, \qquad T(\delta_1) = a_{10}\delta_0 + a_{11}\delta_1,$$
$$T(\delta_n) = k_n\delta_n, \qquad n \geqq 2,$$

where (a_{ij}), $i, j=0, 1$ are complex numbers with moduli not exceeding K and $|k_n| \leqq K^n$ for $n\geqq 2$.

PROOF. The details of the proof are on the same lines as that of Theorem 2 above. In the notation of that proof, the inequalities obtained for $|a_{np}|$ remain valid with $K|c|$ instead of $|c|$ on the right-hand side and as before by letting $c\to 0$ or ∞ we get the desired result. Only, the more precise information derived for isometry will not be valid here.

4. **The algebra** $\Gamma(N)$. If for $\alpha, \beta\in\Gamma$, we define $\alpha\beta=\alpha(z)\beta(z)$, the natural multiplication thus defined endows Γ with the structure of an algebra. Using the equivalence of convergence in Γ with uniform convergence in every finite circle, it follows that the multiplication thus defined is continuous in the topology of Γ. We thus get a topological algebra which we denote by $\Gamma(N)$. We have already proved in [**3**, Theorem 8], that every homomorphism T of $\Gamma(N)$ into itself is of the form $T(\alpha)=\alpha[\beta(z)]$ for every $\alpha\in\Gamma(N)$ where $T(\delta_1)=\beta=\beta(z)$, and conversely. Moreover, T is one-to-one except in the case when $\beta(z)$ is a constant. Also, T is a homomorphism of $\Gamma(N)$ *onto* itself if and only if $\beta(z)=az+b$ where a and b are complex numbers with $a\neq 0$. It is obvious that in this case T is an automorphism of $\Gamma(N)$. The converse statement being obviously true, we see that *the group*

of automorphisms of $\Gamma(N)$ *is isomorphic to the group of one-to-one conformal transformations of the complex plane onto itself leaving the point at infinity invariant.* The algebra $\Gamma(N)$ is commutative and has δ_0 for the identity element.

5. **The algebra** $\Gamma(C)$. We can define multiplication in Γ by using Hadamard composition. If $\alpha = \sum a_n\delta_n$, $\beta = \sum b_n\delta_n$ be two elements of Γ we define $\alpha \circ \beta = \sum a_n b_n \delta_n$. From definition we have $|\alpha \circ \beta| \leqq |\alpha|\,|\beta|$ so that the multiplication thus defined is continuous in the metric topology of Γ. Thus Γ is endowed with the structure of a commutative topological algebra. It has no identity element. We shall denote this algebra by $\Gamma(C)$. For the homomorphisms of $\Gamma(C)$ into itself we prove the following theorem.

THEOREM 5. *With every homomorphism* T *of* $\Gamma(C)$ *into itself we can associate a unique sequence* H_n, $n=0, 1, 2, \cdots$, *of disjoint finite subsets (possibly empty) of the set* I *of non-negative integers having the following properties:*

(i) $T(\delta_n) = \sum_{p\in H_n} \delta_p$, *the right side being interpreted as* 0 *(zero element) when* H_n *is empty;*

(ii) *if* p_n *is the greatest integer in* H_n, *then* $p_n = O(n)$, *as* $n \to \infty$, *writing* $p_n = 0$ *whenever* H_n *is empty. Under these conditions,* $T(\alpha) = \sum a_n T(\delta_n)$ *for all* $\alpha = \sum a_n\delta_n \in \Gamma(C)$. *Conversely if the disjoint finite (possibly empty) subsets of* I *be pre-assigned satisfying* (ii), $T(\delta_n)$ *and* $T(\alpha)$ *be defined as above,* T *is a homomorphism of* $\Gamma(C)$ *into itself.*

PROOF. The converse part follows easily if we observe that T as defined in the theorem is a continuous linear transformation in virtue of (ii) and Theorem 1. Since the sets H_n are disjoint, it can be directly verified that $T(\alpha \circ \beta) = T(\alpha) \circ T(\beta)$. So T is a homomorphism of $\Gamma(C)$ into itself. For the direct part, it is enough to prove (i) and (ii) since the formula for $T(\alpha)$ will follow from the continuity of T. Let $T(\delta_n) = \sum a_{np}\delta_p$. Using the fact that $T(\delta_m \circ \delta_n) = T(\delta_m) \circ T(\delta_n) = 0$ when $m \neq n$ and $= T(\delta_m)$ when $m = n$ we get the equations $a_{mp}a_{np} = 0$ for all $p \geqq 0$ when $m \neq n$ and $a_{mp}^2 = a_{mp}$ for all $p \geqq 0$ and for every $m \geqq 0$. The second equation shows that for a fixed m, $a_{mp} = 0$ or 1, $p \geqq 0$. Since $T(\delta_m)$ is an integral function, it follows that only a finite number (possibly none) of the a_{mp} can be equal to 1, the rest being zero. The first equation shows that if for a particular p and m we have $a_{mp} = 1$, then, for that p, $a_{np} = 0$ for all $n \neq m$. So the matrix (a_{mp}) has the property that each row contains at most a finite number of ones, the rest being zero and no column contains more than one 1. This is precisely the situation reflected in (i). If there are only a finite

number of the sets H_n which are not empty, (ii) is trivial. Otherwise, since T is continuous, Theorem 1 shows that p_n/n must be bounded. This proves (ii) and completes the proof of the theorem.

5.1. THEOREM 6. *Let T be a homomorphism of $\Gamma(C)$ into itself. Let H_n be as in Theorem 5. Then the following statements are true*:

(1) *T maps $\Gamma(C)$ onto $T[\Gamma(C)]$ in a one-to-one manner if and only if no H_n is empty or, equivalently, no $T(\delta_n)$ is zero.*

(2) *If, in addition, $n=O(p_n)$, T is a closed transformation, that is transforms closed sets of $\Gamma(C)$ into closed sets. In particular $T[\Gamma(C)]$ is closed in $\Gamma(C)$ and so is complete and, consequently, T is an isomorphism (one-one and bi-continuous) of $\Gamma(C)$ onto $T[\Gamma(C)]$.*

PROOF. (1) If some $T(\delta_n)$ be zero, T obviously cannot be one-to-one. On the other hand, suppose that no $T(\delta_n)$ is zero. Let $\alpha=\sum a_n\delta_n$, $\alpha=\sum b_n\delta_n$ and $T(\alpha)=T(\beta)$. Then $\sum a_nT(\delta_n)=\sum b_nT(\delta_n)$ and since each $T(\delta_n)$ is the sum of a finite number of terms of the type z^p, it follows that $a_n=b_n$ for all $n\geqq 0$ and so $\alpha=\beta$. This proves (1).

(2) Let E be a closed subset of $\Gamma(C)$ and let α' be an element of the closure of $T(E)$. Then, there exists a sequence $\alpha_p\in E$, $p=1, 2, \cdots$, such that $T(\alpha_p)\to\alpha'$ as $p\to\infty$. Writing $\alpha_p=\sum a_{pn}\delta_n$, we see that the sequence $\sum a_{pn}T(\delta_n)$, $p=1, 2, \cdots$, of integral functions converge uniformly in any finite circle to α'. This implies that for every $R>0$ and given $\epsilon>0$, we can find P such that for all $p, q\geqq P$ $\cdot\sum|a_{pn}-a_{qn}|\,|T(\delta_n);\ R|\leqq\epsilon$. There exists a positive d such that $p_n\geqq dn$. So we get that $\sum|a_{pn}-a_{qn}|R^{dn}\leqq\epsilon$ for $p, q\geqq P$. Such a relation being true for every $R>0$, it follows that the sequence α_p of integral functions converges uniformly to a function α. In other words, $\alpha_p\to\alpha$ as $p\to\infty$ in the topology of $\Gamma(C)$. Since T is continuous, we have $T(\alpha_p)\to T(\alpha)=\alpha'$. But α belongs to E since E is closed. So α' belongs to $T(E)$ and therefore $T(E)$ is closed. Thus T is a closed transformation. In particular, $T[\Gamma(C)]$ is closed in $\Gamma(C)$ and hence complete. Since T maps $\Gamma(C)$ *onto* $T[\Gamma(C)]$ in a one-to-one manner, it follows from a known theorem [**4**, p. 41, Theorem 5] that the inverse of T is also continuous and so T is an isomorphism of $\Gamma(C)$ onto $T[\Gamma(C)]$. This completes the proof of (2).

5.2. THE GROUP OF AUTOMORPHISMS OF $\Gamma(C)$. To describe the group of automorphisms of $\Gamma(C)$, we require a few definitions. We shall denote by $P(I)$ the group of all permutations of the set I of non-negative integers. If $\theta: n\to\theta(n)$ be an element of $P(I)$, consider the following property of θ:

$$\theta(n) = O(n) \quad \text{and} \quad n = O[\theta(n)].$$

The class of all permutations with the above property forms a subgroup $G(I)$ of $P(I)$. It may be noted that $G(I)$ includes every permutation which leaves invariant all but a finite number of elements of I. These latter form an invariant subgroup of $P(I)$ and therefore of $G(I)$. We can now state

THEOREM 7. *The group of automorphisms of* $\Gamma(C)$ *is isomorphic to the group* $G(I)$.

PROOF. Firstly, a permutation θ of $G(I)$ defines an automorphism if we write $T(\delta_n)=\delta_N$ where $N=\theta(n)$. By the previous theorem (2), T defines an isomorphism of $\Gamma(C)$ onto $T[\Gamma(C)]$. So we have to show that $T[\Gamma(C)]=\Gamma(C)$. If $\alpha=\sum a_n\delta_n\in\Gamma(C)$, then $\alpha=T(\alpha')$ where $\alpha'=\sum a_n\delta_{n'}$ and $\theta(n')=n$. So T is an automorphism of $\Gamma(C)$.

Now let T be an automorphism of $\Gamma(C)$. By (1) of the previous theorem, no H_n is empty. No H_n can contain more than one integer of I because if some H_n contained more than one, $T[\Gamma(C)]$ cannot contain integral functions whose Taylor expansions around $z=0$ have no two coefficients equal. Again, every integer in I must be in some H_n for, if an integer p is not in any H_n, $T[\Gamma(C)]$ cannot contain any integral function for which the coefficient of z^p is not zero. The correspondence thus set up between n and the unique integer in H_n is therefore a permutation θ for which $\theta(n)=O(n)$ by Theorem 5 (ii). The same argument applied to the inverse of T shows that $n=O[\theta(n)]$. The correspondence thus set up between an automorphism T of $\Gamma(C)$ and the permutation defined by the sets H_n is obviously an isomorphism between the group of automorphisms of $\Gamma(C)$ and the group $G(I)$.

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